Math 2130 - Assignment 8

Due October 22, 2021

(1) Give 2 different bases for

$$H = \text{Span}\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}3\\-1\\0\end{bmatrix}, \begin{bmatrix}-1\\3\\4\end{bmatrix}\}$$

Solution:
Row reduction yields

$$\begin{bmatrix}1 & 3 & -1\\1 & -1 & 3\\2 & 0 & 4\end{bmatrix} \sim \begin{bmatrix}1 & 3 & -1\\0 & -4 & 4\\0 & 0 & 0\end{bmatrix}.$$

So the first 2 columns of the original matrix are pivot columns and form a basis of H.

$$B_1 = (\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}3\\-1\\0\end{bmatrix})$$

Recall that the order of basis vectors is important (coordinates!). So by flipping the vectors we get a different basis

$$B_2 = (\begin{bmatrix}3\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix})$$

Also note that any 2 columns of the matrix are linearly independent. So any 2 distinct vectors in any order out of $\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}3\\-1\\0\end{bmatrix}, \begin{bmatrix}-1\\3\\4\end{bmatrix}\}$ form a basis. $\square$

(2) Let $B = (b_1, \dots, b_n)$ be a basis for a vector space V and consider the coordinate mapping $V \to \mathbb{R}^n,\ x \mapsto [x]_B$.
 (a) Show that $[c \cdot x]_B = c[x]_B$ for all $x \in V, c \in \mathbb{R}$.
 (b) Show that the coordinate mapping is onto $\mathbb{R}^n$.
Solution:

 (a) Let $x \in V$ with $x = c_1 b_1 + \dots c_n b_n$ for $c_1, \dots, c_n \in \mathbb{R}$. That is, $[x]_B = \begin{bmatrix}c_1\\\vdots\\c_n\end{bmatrix}$.

 Let $c \in \mathbb{R}$ and consider

$$cx = c(c_1 b_1 + \dots c_n b_n) = cc_1 b_1 + \dots cc_n b_n.$$

 Then the coordinates of cx are

$$[cx]_B = \begin{bmatrix}cc_1\\\vdots\\cc_n\end{bmatrix} = c\begin{bmatrix}c_1\\\vdots\\c_n\end{bmatrix} = c[x]_B.$$

(b) To show the map is onto $\mathbb{R}^n$, let $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$. We have to find $x \in V$ such that $[x]_B = y$. Pick $x = y_1 b_1 + \dots y_n b_n$. This shows that the coordinate map is onto. $\square$

(3) Let $B = (\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix})$ be a basis of $\mathbb{R}^2$.

(a) Give the change of coordinates matrix $P_{E \leftarrow B}$ from B to the standard basis $E = (e_1, e_2)$ and $P_{B \leftarrow E}$.

(b) Find vectors $u, v \in \mathbb{R}^2$ with $[u]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [v]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

(c) Compute the coordinates relative to B of $w = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution:

(a) For $P_{E \leftarrow B}$ we put the vectors of B in the columns of a matrix,

$$P_{E \leftarrow B} = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$$

(b) $u = P_{E \leftarrow B} \cdot [u]_B = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, the second vector in B

$v = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

(c) To find $[w]_B$ we can solve $P_{E \leftarrow B} \cdot [w]_B = w$ directly by row reduction. Alternatively, we can invert $P_{E \leftarrow B}$ and use the formula $[w]_B = P_{E \leftarrow B}^{-1} \cdot w$.

$$P_{E \leftarrow B}^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

So

$$[w]_B = \frac{1}{-2} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$[x]_B = \frac{1}{-2} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$\square$

(4) Let $B = (1, t, t^2)$ and $C = (1, 1+t, 1+t+t^2)$ be bases of $\mathbb{P}_2$.

(a) Determine the polynomials p, q with $[p]_B = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$ and $[q]_C = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$.

(b) Compute $[r]_B$ and $[r]_C$ for $r = 3 + 2t + t^2$.

Solution:

(a) $p = 3 - 2t^2$,
$q = 3 \cdot 1 + 0 \cdot (1+t) - 2(1+t+t^2) = 1 - 2t - 2t^2$

(b) For the coordinates relative to B just take the coefficients of the polynomial:

$[r]_B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

For the coordinates relative to C consider the equation

$$\begin{aligned} r &= x_1 \cdot 1 + x_2(1+t) + x_3(1+t+t^2) \\ &= (x_1+x_2+x_3) + (x_2+x_3)t + x_3 t^2 \end{aligned}$$

Comparing the coefficients we obtain $x_3 = 1, x_2 = 1, x_1 = 1$. So $[r]_C = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ □

(5) Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 1 \\ 2.5 \\ -5 \end{bmatrix}$.

(a) Find vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ such that $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{u}_1, \ldots, \mathbf{u}_k)$ is a basis for $\mathbb{R}^3$.

(b) Find vectors $\mathbf{v}_1, \ldots, \mathbf{v}_\ell$ such that $(\mathbf{b}_3, \mathbf{v}_1, \ldots, \mathbf{v}_\ell)$ is a basis for $\mathbb{R}^3$.

Prove that your choices for (a) and (b) form a basis.

Solution:

Both bases have 3 vectors. Thus $k = 1$ and $\ell = 2$.

(a) One possible choice is $\mathbf{u}_1 = \mathbf{e}_1$. We show that $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{e}_1)$ is a basis by reducing the following augmented matrix to echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

There is no free variable. Thus the columns are linearly independent. Also, the vectors span $\mathbb{R}^3$ since there is no zero row.

(b) One possible choice is $\mathbf{v}_1 = \mathbf{e}_1$ and $\mathbf{v}_2 = \mathbf{e}_2$. We show that $(\mathbf{b}_3, \mathbf{e}_1, \mathbf{e}_2)$ is a basis by reducing the following augmented matrix to echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2.5 & 0 & 1 & 0 \\ -5 & 0 & 0 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

There is no free variable. Thus the columns are linearly independent. Also, the vectors span $\mathbb{R}^3$ since there is no zero row.

□

(6) Let

$$A = \begin{bmatrix} -5 & 8 & 0 & -17 & -2 \\ 3 & -5 & 1 & 5 & 1 \\ 11 & -19 & 7 & 1 & 3 \\ 7 & -13 & 5 & -3 & 1 \end{bmatrix}.$$

Find bases and dimensions for $\operatorname{Nul} A$, $\operatorname{Col} A$, and $\operatorname{Row} A$, respectively.

Solution:

We reduce A to reduced echelon form:

$$A \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 5 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For $\operatorname{Nul} A$, we solve $A\mathbf{x} = \mathbf{0}$ and obtain

$$\operatorname{Nul} A = \{r \begin{bmatrix} -5 \\ -1 \\ 5 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mid r, s \in \mathbb{R}\}.$$

The two vectors form a basis for $\operatorname{Nul} A$.

The first three columns of A contain a pivot. Thus they form a basis

$$B = (\begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 8 \\ -5 \\ -19 \\ -13 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix})$$

for $\operatorname{Col} A$.

The nonzero rows in any echelon form of A form a basis. E.g.,

$$C = (\begin{bmatrix} 1 \\ 0 \\ 0 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -5 \\ 0 \end{bmatrix})$$

is a basis for $\operatorname{Row} A$. □

(7) A 25×35 matrix A has 20 pivots. Find $\dim \operatorname{Nul} A$, $\dim \operatorname{Col} A$, $\dim \operatorname{Row} A$, and $\operatorname{rank} A$.

Solution:

The number of pivots, $\dim \operatorname{Row} A$, $\dim \operatorname{Col} A$, and the rank are equal. So

$$\dim \operatorname{Row} A = \dim \operatorname{Col} A = \operatorname{rank} A = 20.$$

By the rank theorem, $\dim \operatorname{Nul} A + \operatorname{rank} A = 35$. Thus

$$\dim \operatorname{Nul} A = 35 - 20 = 15.$$

□

(8) True or false? Explain.

(a) If B is an echelon form of a matrix A, then the pivot columns of B form a basis for the column space of A.

(b) If B is an echelon form of a matrix A, then the nonzero rows of B form a basis for the row space of A.

(c) A basis of B is a set of linear independent vectors in V that is as large as possible.

(d) If $\dim V = n$, then any n vectors that span V are linearly independent.

(e) Every 2-dimensional subspace of $\mathbb{R}^2$ is a plane.

Solution:

(a) False! In general the columns of B will not span $\operatorname{Col} A$ any more. The pivot columns of A form a basis for $\operatorname{Col} A$.

(b) True. The rows of B still span $\operatorname{Row} A$.

(c) True. If linear independent vectors $a_1, \ldots, a_k$ in V do not span V yet, you can get a bigger linear set by adding a vector a_{k+1} which is not in $\operatorname{Span}\{a_1, \ldots, a_k\}$.

(d) True by the Basis Theorem.
(e) True since any two linear independent vectors in $\mathbb{R}^2$ span a plane through the origin.

$\square$