Math 2130 - Assignment 6

Due October 8, 2021

(1) Compute the inverse if possible:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 2 & 4 \\ 0 & 1 & -2 \\ 1 & -3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ -1 & 2 & -1 \end{bmatrix}$$

(2) Let $A, B \in \mathbb{R}^{n\times n}$ be invertible. Show $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$.

(3) A matrix $C \in \mathbb{R}^{n\times m}$ is called a **left inverse** of a matrix $A \in \mathbb{R}^{m\times n}$ if $CA = I_n$ (the $n \times n$ identity matrix).

  (a) Show that if A has a left inverse C, then $Ax = b$ has a unique solution for any $b \in \mathbb{R}^n$.

  (b) Give an example of a matrix A that has a left inverse but is not invertible.

(4) Prove that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is not invertible if $ad - bc = 0$.

Hint: Show that the columns of A are linearly dependent. Consider the cases $a = 0$ and $a \neq 0$ separately.

(5) Let A be an **upper triangular matrix**, that is,

$$A = \begin{bmatrix} a_{11} & \dots & \dots & a_{1n} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a_{nn} \end{bmatrix}$$

with zeros below the diagonal. Show

  (a) A is invertible iff there are no zeros in the diagonal of A.

  (b) If A^{-1} exists, it is an upper triangular matrix as well.

  Hint: When row reducing $[A, I_n]$ to $[I_n, A^{-1}]$, what happens to the n columns on the right?

(6) Assume that $f\colon \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto A \cdot x$, is bijective. Show that $A \in \mathbb{R}^{n\times n}$ is invertible.

Give a formula for the inverse function f^{-1}.

Hint: Use that f is surjective and the Invertible Matrix Theorem.

(7) (a) What is the inverse of the rotation R by angle α counter clockwise around the origin in $\mathbb{R}^2$? What is the standard matrix of R^{-1}?

  (b) What is the inverse of a reflection S on a line through the origin in $\mathbb{R}^2$? What can you say about the standard matrix B of S and its inverse? You do not have to write down B for this.

(8) True of false? Explain your answer.

  (a) If A, B are square matrices with $AB = I_n$, then A and B are invertible.

  (b) If A is invertible, then A^T is invertible.

  (c) Let $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ such that $Ax = b$ is inconsistent. Then $\mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax$ is not injective.