Math 2001 - Assignment 10

Due April 3, 2020

(1) [1, Chapter 10, exercise 8] Show that for every $n \in \mathbb{N}$:
$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

(2) Let $n \in \mathbb{N}$ and let $A_1, \ldots, A_n$ be sets in some universe U. Show by induction that
$$\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \bar{A}_1 \cap \bar{A}_2 \cap \ldots \bar{A}_n$$
Hint: for the base case consider $n = 1$ and $n = 2$.

(3) Show by induction that for every natural number $n \geq 4$:
$$2^n \geq n^2$$

(4) Define a sequence of integers $a_1 := 1, a_2 := 1$ and
$$a_n := 2a_{n-1} + a_{n-2} \text{ for } n \geq 3.$$
Prove that a_n is odd for all $n \in \mathbb{N}$ by strong induction.

(5) Let $p_1, p_2, \ldots$ denote the list of all primes. Show that for integers $a = \Pi_{i \in \mathbb{N}}\, p_i^{e_i}, b = \Pi_{i \in \mathbb{N}}\, p_i^{f_i}$ with $e_i, f_i \in \mathbb{N}_0$ for $i \in \mathbb{N}$,
$$\operatorname{lcm}(a, b) = \Pi_{i \in \mathbb{N}}\, p_i^{\max(e_i, f_i)}.$$

(6) Show for all $a, b \in \mathbb{N}$:
$$\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$$
Hint: Use the formula for gcd and lcm from class and the previous problem.

References

[1] Richard Hammack. The Book of Proof. Creative Commons, 3rd edition, 2018. Available for free: `http://www.people.vcu.edu/~rhammack/BookOfProof/`