Math 2001 - Assignment 7

Due March 6, 2020

Be careful to write down every step in the proofs of problems 4,5 and reduce every statement to definitions or other statements that were already proved in class.

(1) How many 4-letter "words" can you form from the alphabet $A, \dots, Z$ if the letters are in alphabetical order and
  (a) repetitions are not allowed, e.g., BEFS,
  (b) repetitions are allowed, e.g., BFFS.

(2) In how many different ways can n students (called $A, B, ...$) line up in a queue for the cafeteria such that
  (a) students A and B stand next to each other?
  (b) students A and B do not stand next to each other?

(3) How many different seating arrangements are there on a round table with n seats?

Hint: Since a round table has no beginning or end, two arrangements are the same if one is obtained from the other by rotation, e.g., the following are considered equal:

Represent seating arrangements by lists. When do 2 lists describe the same arrangement?

(4) $a \in \mathbb{Z}$ is *even* if $a = 2n$ for some $n \in \mathbb{Z}$.
$a \in \mathbb{Z}$ is *odd* if $a = 2n + 1$ for some $n \in \mathbb{Z}$.

Prove:
  (a) Let $a \in \mathbb{Z}$. If a is even, then a^2 is even.
  Hint: Use a direct proof. Assume a is a multiple of 2. Show that a^2 is a multiple of 2.
  (b) If x is an odd integer, then 8 divides $x^2 - 1$.
  Hint: Use a direct proof. Then split into 2 cases.

(5) Complete the proof from class that $\gcd(a, b) = \gcd(a - qb, b)$ for all $a, b, q \in Z$ with not both a and b equal 0.

Assume $d|a - qb$ and $d|b$. Show that $d|a$ and $d|b$.

(6) Compute $\gcd(a, b)$ and the Bezout coefficients using the Euclidean algorithm for the following numbers. Then find $\mathrm{lcm}(a, b)$.
  (a) $a = 85, b = 25$
  (b) $a = 57, b = 24$