Review 2: Logic

Peter Mayr

CU, Discrete Math, April 27, 2020

Implications

Let P, Q be statements.

statement	equivalent meanings	negation
$P \Rightarrow Q$	If P is true, then Q is true. $\sim P \vee Q$ $\sim Q \Rightarrow \sim P$ (contrapositive)	$P \wedge \sim Q$
$P \Leftrightarrow Q$	Q is true if and only if P is true. $(P \Rightarrow Q) \wedge (P \Leftarrow Q)$	$P \Leftrightarrow \sim Q$ $\sim P \Leftrightarrow Q$

How to prove $P \Rightarrow Q$:

- **direct:** Assume P. Show Q.
- **contrapositive:** Assume $\sim Q$. Show $\sim P$.
- **by contradiction** (only as last resort): Assume P and $\sim Q$. Show some contradiction.

How to prove $P \Leftrightarrow Q$:

Show $P \Rightarrow Q$ and show $P \Leftarrow Q$.

Equivalence: an application in Linear Algebra

Theorem

Let A be an $n \times n$-matrix over $\mathbb{R}$. Then the following are equivalent (TFAE):

1. A is invertible.
2. $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$.
3. $Ax = 0$ has only the trivial solution.
4. $\det A \neq 0$.
5. 0 is not an eigenvalue of A.

This Theorem states that 1. $\Leftrightarrow$ 2. $\Leftrightarrow$ 3. $\Leftrightarrow$ 4. $\Leftrightarrow$ 5.
Instead of showing 4 times $\Leftrightarrow$, such statements can be proved more efficiently as a cycle of implications:

$$1. \Rightarrow 2. \Rightarrow 3. \Rightarrow 4. \Rightarrow 5. \Rightarrow 1.$$

- Do you want to know more? Take Math 2135 – Linear Algebra

Quantified statements

Let A be a set, $P(x)$ be a statement for $x \in A$.

statement	meaning	negation
$\forall x \in A \colon P(x)$	For all $x \in A$, $P(x)$ is true.	$P(x)$ is not true for some $x \in A$. $\exists x \in A \colon \ \sim P(x)$
$\exists x \in A \colon P(x)$	There exists $x \in A$ such that $P(x)$ is true.	$P(x)$ is not true for all $x \in A$. $\forall x \in A \colon \ \sim P(x)$

How to prove $\forall x \in A \colon P(x)$

Let $x \in A$ (arbitrary but fixed). Show $P(x)$.

How to refute $\forall x \in A \colon P(x)$

Give concrete explicit $x \in A$ that does not satisfy $P(x)$.

How to prove $\exists x \in A \colon P(x)$

Give a concrete explicit $x \in A$ that satisfies $P(x)$.

How to refute $\exists x \in A \colon P(x)$

Show $\forall x \in A \colon \ \sim P(x)$.

Quantifiers and implications in Calculus

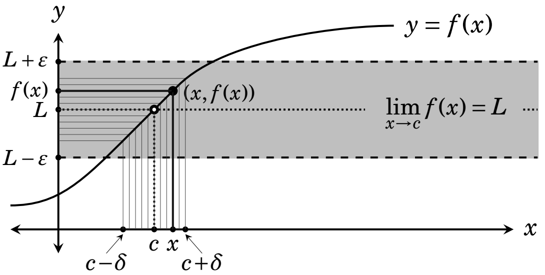
Calculus: Informally $\lim_{x\to c} f(x) = L$ means that $f(x)$ is arbitrarily close to L provided that x is sufficiently close to c. More precisely

Definition (Limit of a function)

Let $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. Then $\lim_{x\to c} f(x) = L$ if

$$\forall \varepsilon \in \mathbb{R}^+ \; \exists \delta \in \mathbb{R}^+ \; \forall x \in \mathbb{R} - \{c\} : \; |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Diagram taken from Hammack, Book of Proof, 2018.



How to prove $\forall \varepsilon \in \mathbb{R}^+ \ \exists \delta \in \mathbb{R}^+ \ \forall x \in \mathbb{R}: \ |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Example

Prove $\lim_{x \to 0} 2x \sin(\frac{1}{x}) = 0$.

- Let $\varepsilon > 0$ arbitrary, fixed for the remainder of the proof.
- Goal: Find $\delta > 0$ (δ may depend on ε) such that $|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$.
- Consider

$$|f(x) - L| = \left| 2x \sin(\frac{1}{x}) - 0 \right| = 2|x| \underbrace{|\sin(\frac{1}{x})|}_{\leq 1} \leq 2|x|$$

- The latter is $< \varepsilon$ if $|x| < \varepsilon/2$.
- Hence $\delta := \varepsilon/2$ yields $|x - 0| < \delta \Rightarrow |f(x) - 0| < \varepsilon$.
- Thus $\lim_{x \to 0} 2x \sin(\frac{1}{x}) = 0$.

How to prove a limit does not exist?

We need to show that for all $L \in \mathbb{R}$,

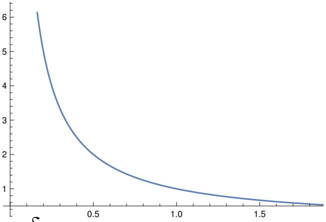
$$\sim (\forall \varepsilon \in \mathbb{R}^+ \, \exists \delta \in \mathbb{R}^+ \, \forall x \in \mathbb{R} - \{c\} : \; |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon),$$

equivalently,

$$\forall L \in \mathbb{R} \, \exists \varepsilon \in \mathbb{R}^+ \, \forall \delta \in \mathbb{R}^+ \, \exists x \in \mathbb{R} - \{c\} : \; |x - c| < \delta \text{ and } |f(x) - L| \geq \varepsilon.$$

Example

Prove $\lim_{x \to 0} \frac{1}{x}$ does not exist.



- Let $L \in \mathbb{R}$, fix say $\varepsilon := 1$, let δ be arbitrary.
- Want $x \in \mathbb{R}$: $|x - 0| < \delta$ and $|\frac{1}{x} - L| \geq \varepsilon$.
- E.g., let $n \in \mathbb{N}$ such that $x := \frac{1}{L+n} < \delta$. Then $|f(x) - L| = |(L + n) - L| = n \geq 1$.
- Thus $\lim_{x \to 0} \frac{1}{x}$ cannot be any $L \in \mathbb{R}$.

Sum rule for limits

Theorem

If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then

$$\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x).$$

Proof.

- Let $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Show $\lim_{x\to c}(f(x)+g(x)) = L+M$.
- Let $\varepsilon > 0$ arbitrary. Find $\delta > 0$ such that $|x-c| < \delta \Rightarrow |f(x)+g(x)-L-M| < \varepsilon$.
- Note $|f(x)+g(x)-L-M| \leq |f(x)-L| + |g(x)-M|$.
- Since $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, we have
  $\delta_1 > 0$ such that $|x-c| < \delta_1 \Rightarrow |f(x)-L| < \frac{\varepsilon}{2}$,
  $\delta_2 > 0$ such that $|x-c| < \delta_2 \Rightarrow |g(x)-M| < \frac{\varepsilon}{2}$.
- Then $|x-c| < \underbrace{\min(\delta_1,\delta_2)}_{=:\delta} \Rightarrow |f(x)+g(x)-L-M| < \underbrace{\frac{\varepsilon}{2}+\frac{\varepsilon}{2}}_{=\varepsilon}$.

- Do you want to know more? Take Math 3001 – Analysis