Induction

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Example

Compute the sum of the first n odd positive integers:

$$\begin{array}{lll} n & & \\ 1 & 1 & = 1 \\ 2 & 1+3 & = 4 \\ 3 & 1+3+5 & = 9 \\ 4 & 1+3+5+7 & = 16 \\ \vdots & & \end{array}$$

Conjecture

$1 + 3 + 5 + \cdots + (2n - 1) = n^2$

Question

How to prove this?

Let S_n be a statement (depending on $n \in \mathbb{N}$), e.g., $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Theorem

S_n is true for all $n \in \mathbb{N}$.

Induction proof.

1. **basis step:** Show S_1.
2. **inductive step:** Show $S_k \Rightarrow S_{k+1}$ for any $k \in \mathbb{N}$.

This shows that S_n is true for every $n \in \mathbb{N}$ because:

By basis step, S_1 is true.

By inductive step,

$$\begin{aligned} &S_1 \Rightarrow S_2;\ \text{so } S_2 \text{ is true.}\\ &S_2 \Rightarrow S_3;\ \text{so } S_3 \text{ is true.}\\ &S_3 \Rightarrow S_4\\ &\qquad\vdots \end{aligned}$$

By this domino effect, all S_n are true!

Theorem

For all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n}(2i-1) = n^2.$$

Proof (by induction)

- **Basis step, $n = 1$:** $2 \cdot 1 - 1 = 1^2$ holds
- **Induction assumption, S_k holds for some fixed k:** $\sum_{i=1}^{k}(2i-1) = k^2$
- **Induction step, S_{k+1} follows:**

$$\begin{aligned} \sum_{i=1}^{k+1}(2i-1) &= \sum_{i=1}^{k}(2i-1) + (2k+1) \\ &= k^2 + 2k + 1 \text{ by induction assumption} \\ &= (k+1)^2 \end{aligned}$$

This proves $S_k \Rightarrow S_{k+1}$, hence the Theorem $\square$

If induction is not strong enough for you any more ...

Binomial Theorem

For all $n \in \mathbb{N}_0$,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

Proof (by induction on n)

- **Basis step, $n = 0$:** $(x+y)^0 = 1 = \binom{0}{0} x^0 y^0$ holds
- **Induction assumption, formula holds for fixed $n = k$:**
- **Induction step, show the formula for $n = k+1$ follows:**

$$\begin{aligned}
& (x+y)^{k+1} \\
= \;& (x+y)^k \cdot (x+y) \\
= \;& \left[\tbinom{k}{0} x^k y^0 + \tbinom{k}{1} x^{k-1} y^1 + \tbinom{k}{2} x^{k-2} y^2 + \cdots + \tbinom{k}{k} x^0 y^k \right] (x+y) \\
= \;& \tbinom{k}{0} x^{k+1} y^0 + \tbinom{k}{1} x^k y^1 + \tbinom{k}{2} x^{k-1} y^2 + \cdots + \tbinom{k}{k} x^1 y^k \\
& \qquad + \tbinom{k}{0} x^k y^1 + \tbinom{k}{1} x^{k-1} y^2 + \cdots + \tbinom{k}{k-1} x^1 y^k + \tbinom{k}{k} x^0 y^{k+1} \\
= \;& x^{k+1} y^0 + \tbinom{k+1}{1} x^k y^1 + \tbinom{k+1}{2} x^{k-1} y^2 + \cdots + \tbinom{k+1}{k} x^1 y^k + x^0 y^{k+1}
\end{aligned}$$

Induction proof of Binomial Theorem concluded.

We showed the induction step

$$(x+y)^{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} x^{k+1-i} y^i.$$

Hence the Binomial Theorem is proved for all $n \in \mathbb{N}$. $\square$

Theorem (Bernoulli's inequality)

For all $n \in \mathbb{N}$ and for all $x \in \mathcal{R}$ with $x > -1$,

$$(1+x)^n \geq 1 + nx.$$

Proof (by induction on n)

- **Basis step, $n = 1$:** $(1+x)^1 \geq 1 + 1x$ holds
- **Induction assumption for fixed k:** $(1+x)^k \geq 1 + kx$
- **Induction step:** Show $(1+x)^{k+1} \geq 1 + (k+1)x$.
  Since $1 + x > 0$, multiplying the induction assumption by $1 + x$ yields

$$\begin{aligned} (1+x)^{k+1} &\geq (1+kx)(1+x) \\ &= 1 + kx + x + kx^2 \\ &= 1 + (k+1)x + \underbrace{kx^2}_{\geq 0} \end{aligned}$$

Thus $(1+x)^{k+1} \geq 1 + (k+1)x$. $\square$