Cardinality of sets, 4

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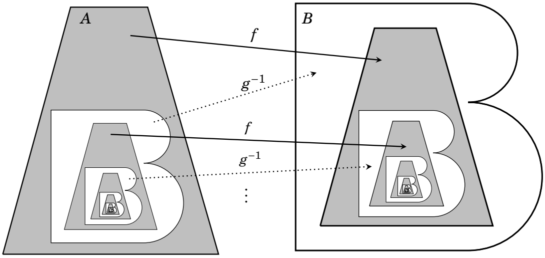
Theorem (Schröder-Bernstein)

Let $f\colon A \to B$ and $g\colon B \to A$ be injective. Then there exists a bijection $h\colon A \to B$.

Proof

The gray area on the left is $G := \bigcup_{k\in\mathbb{N}_0} (g \circ f)^k \, (A - g(B))$
$= (A - g(B)) \;\cup\; (g \circ f)\,(A - g(B)) \;\cup\; (g \circ f)^2\,(A - g(B)) \;\cup \ldots$



Claim:

$$h\colon A \to B,\; x \mapsto \begin{cases} f(x) & \text{if } x \in G, \\ g^{-1}(x) & \text{if } x \in W, \end{cases} \quad \text{is bijective.}$$

$G := \bigcup_{k\in\mathbb{N}_0} (g \circ f)^k (A - g(B))$
$W := A - G$

$$h\colon A \to B,\ x \mapsto \begin{cases} f(x) & \text{if } x \in G, \\ g^{-1}(x) & \text{if } x \in W. \end{cases}$$

For injectivity, let $x, y \in A$ such that $h(x) = h(y)$.

- Case $x, y \in G$: Then $f(x) = f(y)$ implies $x = y$ since f is injective.
- Case $x, y \in W$: Then $g^{-1}(x) = g^{-1}(y)$ implies $x = y$ by applying g on both sides.
- Case $x \in G, y \in W$: Then $f(x) = g^{-1}(y)$ implies $y = (g \circ f)(x) \in (g \circ f)(G) \subseteq G$ by the definition of G. Contradiction.

Hence h is injective.

$G := \bigcup_{k\in\mathbb{N}_0}(g\circ f)^k\,(A - g(B))$
$W := A - G$

$$h\colon A \to B,\; x \mapsto \begin{cases} f(x) & \text{if } x \in G, \\ g^{-1}(x) & \text{if } x \in W. \end{cases}$$

For surjectivity, let $y \in B$ and find $x \in A$ such that $h(x) = y$.

- Case $g(y) \in W$: Then $h(\underbrace{g(y)}_{=:x}) = g^{-1}(g(y)) = y$.
- Case $g(y) \in G$: From the definition of G, we have $k \in \mathbb{N}_0$ and $z \in A - g(B)$ such that

$$g(y) = (g\circ f)^k(z).$$

  - $k > 0$ because else $g(y) = z \in A - g(B)$ is a contradiction.
  - Then $y = f \circ \underbrace{(g\circ f)^{k-1}(z)}_{=:x\in G}$ since g is injective.
  - Hence $h(x) = f(x) = y$.

Thus h is surjective. $\square$

We constructed a bijection $A \to B$ by patching together injections $A \to B$ and $B \to A$.

$|P(\mathbb{N})| = |\mathbb{R}|$

Theorem
$|P(\mathbb{N})| = |\mathbb{R}|$

Proof.
By a previous Thm and Schröder-Bernstein it suffices to construct injections between $P(\mathbb{N})$ and $[0, 1)$:

- Define $g\colon P(\mathbb{N}) \to [0, 1)$ as

$$g(A) := 0.x_1x_2x_3\ldots \text{ in decimal where } x_i := \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{else.} \end{cases}$$

  E.g. $g(\{1, 3\}) = 0.101$
  $g(\{2n \ : \ n \in \mathbb{N}\}) = 0.010101\ldots$ (periodic)

- For $f\colon [0,1) \to P(\mathbb{N})$ consider $x = 0.x_1x_2x_3\dots$ in binary (i.e. $x_i \in \{0,1\}$) and define

$$f(x) := \{i \in \mathbb{N} \ : \ x_i = 1\}.$$

  E.g. $f(0.101) = \{1,3\}$
  $f(0.010101\dots) = \{2n \ : \ n \in \mathbb{N}\}$

  f is injective but not surjective since e.g. $\mathbb{N} \notin f([0,1))$. Note $0.111\cdots = 1$ in binary. $\square$

$|A| < |P(A)|$

Theorem

$|A| < |P(A)|$ for any set A.

Already known for finite A since then $|P(A)| = 2^{|A|}$.

Proof.

- $|A| \leq |P(A)|$ since $g \colon A \to P(A)$, $x \mapsto \{x\}$, is injective.
- To get $|A| < |P(A)|$, show that no $f \colon A \to P(A)$ is surjective (cf. Cantor's diagonal argument).
- Let $f \colon A \to P(A)$ arbitrary and
$$B := \{x \in A \ : \ x \notin f(x)\}.$$
- **Claim:** $f(a) \neq B$ for all $a \in A$.
  - Case $a \notin f(a)$: Then $a \in B$ by definition. Hence $f(a) \neq B$ because else $a \notin B$ and $a \in B$ (contradiction).
  - Case $a \in f(a)$: Then $a \notin B$. Hence $f(a) \neq B$.
- Hence f is not surjective. There is no bijection $A \to P(A)$.

Bigger and bigger

By the previous Theorem, we have a chain of strictly increasing infinite cardinalities,

$$|\mathbb{N}| < \underbrace{|P(\mathbb{N})|}_{=|\mathbb{R}|} < |P(P(\mathbb{N}))| < |P(P(P(\mathbb{N})))| < \dots$$

- $\aleph_0$ (aleph nought) is the cardinality of the least infinite set $\mathbb{N}$.
- The cardinality of $|\mathbb{R}|$ is often called c (for continuum).

The Continuum Hypothesis

Recall $|\mathbb{N}| < |\mathbb{R}|$

Continuum Hypothesis (CH)

There is no set whose cardinality is strictly between $|\mathbb{N}|$ and $|\mathbb{R}|$.

- CH was proposed by Cantor 1878.
- As it turned out, CH can neither be proved nor disproved within Zermelo-Fraenkel Set Theory (ZF).
- CH is **independent** from ZF; true or false depending on what additional axioms you accept to build your sets (Gödel 1940, Cohen 1963).