Cardinality of sets, 3

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Having the same cardinality is an equivalence

Recall

$|A| = |B|$ if there exists a bijection $f\colon A \to B$.

Lemma

Let U be a set (universe) of sets. Having the same cardinality is an equivalence relation on U.

Proof.

- reflexive: Let $A \in U$. Then $|A| = |A|$ by the identity map id_A.
- symmetric: Let $A, B \in U$. If $f\colon A \to B$ is a bijection (i.e., $|A| = |B|$), then $f^{-1}\colon B \to A$ is a bijection, so $|B| = |A|$.
- transitive: Let $A, B, C \in U$. If $f\colon A \to B, g\colon B \to C$ are bijections, then $g \circ f\colon A \to C$ is a bijection, so $|A| = |C|$.

□

Example

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are equivalent (have the same cardinality),
$(0, 1)$ and $\mathbb{R}$ are equivalent.

When is one set smaller than another?

Recall

For finite sets A, B there exists a injective map $f\colon A \to B$ iff $|A| \leq |B|$.

This motivates the following general definition.

Definition

$|A| \leq |B|$ if there exists an injection $f\colon A \to B$.
$|A| < |B|$ if there exists an injection $f\colon A \to B$ but no bijection.

Example

$|\mathbb{N}| \leq |\mathbb{R}$ since $\mathbb{N} \to \mathbb{R},\ x \mapsto x$ is injective.
$|\mathbb{N}| < |\mathbb{R}|$ since there is no bijection $\mathbb{N} \to \mathbb{R}$.

Ordering cardinalities

Lemma

$\leq$ on cardinalities is reflexive and transitive.

Proof.

For transitivity: If $f\colon A \to B$ and $g\colon B \to C$ are injective, then $g \circ f\colon A \to C$ is injective. □

For a partial order relation we also need that $\leq$ is antisymmetric.

Question

If $f\colon A \to B$ and $g\colon B \to A$ are injective, does there exist a bijection $A \to B$?

Yes, for A, B finite.

Example

Can we find a bijection $(-1,1) \to [-1,1]$ from these injections?

$$f\colon (-1,1) \to [-1,1],\ x \mapsto x, \qquad g\colon [-1,1] \to (-1,1),\ x \mapsto \frac{x}{2}$$

Theorem (Schröder-Bernstein)

If there exist injections $f\colon A \to B$ and $g\colon B \to A$, then there exists a bijection $h\colon A \to B$.

Proof sketch.

Diagrams taken from Hammack, Book of Proof, 2018.

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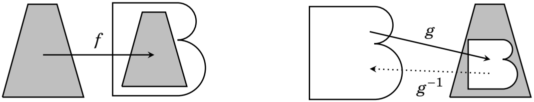
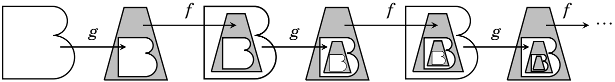


Figure 14.4. The injections $f : A \to B$ and $g : B \to A$

Consider the chain of injections illustrated in Figure 14.5. On the left, g puts a copy of B into A. Then f puts a copy of A (containing the copy of B) into B. Next, g puts a copy of this B-containing-A-containing-B into A, and so on, always alternating g and f.



Folding up the previous chain of injections we get:

Figure 14.6 suggests our desired bijection $h : A \to B$. The injection f sends the gray areas on the left bijectively to the gray areas on the right. The injection $g^{-1} : g(B) \to B$ sends the white areas on the left bijectively to the white areas on the right. We can thus define $h : A \to B$ so that $h(x) = f(x)$ if x is a gray point, and $h(x) = g^{-1}(x)$ if x is a white point.

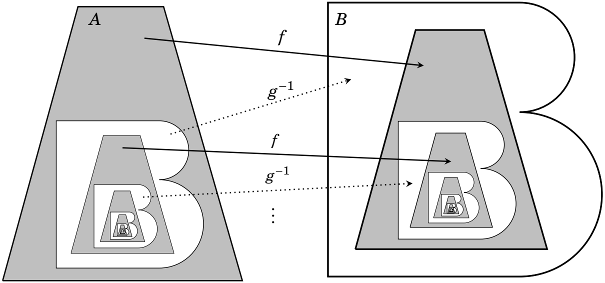


Figure 14.6. The bijection $h : A \to B$

Theorem (Schröder-Bernstein)

Let $f\colon A \to B$ and $g\colon B \to A$ be injective. Then there exists a bijection $h\colon A \to B$.

Proof

- The gray area on the left in Fig 14.6 is $G := \bigcup_{k \in \mathbb{N}_0} (g \circ f)^k \,(A - g(B))$.
- $g\colon B \to g(B)$ is bijective, in particular the inverse g^{-1} exists on $W := A - G$.

Claim:

$$h\colon A \to B,\ x \mapsto \begin{cases} f(x) & \text{if } x \in G, \\ g^{-1}(x) & \text{if } x \in W, \end{cases}$$

is bijective.

$G := \bigcup_{k\in\mathbb{N}_0}(g \circ f)^k\,(A - g(B))$ $\qquad h\colon A \to B,\ x \mapsto \begin{cases} f(x) & \text{if } x \in G, \\ g^{-1}(x) & \text{if } x \in W. \end{cases}$

For injectivity, let $x, y \in A$ such that $h(x) = h(y)$.

- Case $x, y \in G$: Then $f(x) = f(y)$ implies $x = y$ since f is injective.
- Case $x, y \in W$: Then $g^{-1}(x) = g^{-1}(y)$ implies $x = y$ by applying g on both sides.
- Case $x \in G, y \in W$: Then $f(x) = g^{-1}(y)$ implies $y = (g \circ f)(x) \in (g \circ f)(G) \subseteq G$ by the definition of G. Contradiction.

Hence h is injective.

$G := \bigcup_{k\in\mathbb{N}_0}(g \circ f)^k\,(A - g(B))$ $\qquad h\colon A \to B,\ x \mapsto \begin{cases} f(x) & \text{if } x \in G, \\ g^{-1}(x) & \text{if } x \in W. \end{cases}$

For surjectivity, let $y \in B$ and find $x \in A$ such that $h(x) = y$.

- Case $g(y) \in W$: Then $h(\underbrace{g(y)}_{=x}) = g^{-1}(g(y)) = y$.
- Case $g(y) \in G$: From the definition of G, we have $k \in \mathbb{N}_0$ and $z \in A - g(B)$ such that

$$g(y) = (g \circ f)^k(z).$$

  - $k > 0$ because else $g(y) = z \in A - g(B)$ is a contradiction.
  - Then $y = f \circ \underbrace{(g \circ f)^{k-1}(z)}_{=x\in G}$ since g is injective.
  - Hence $h(x) = f(x) = y$.

Thus h is surjective. □