Cardinality of sets 2

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Recall

Definition

Sets A and B have the same **cardinality**, written $|A| = |B|$, if there exists a bijection $f\colon A \to B$.

Definition

A set A is **finite** if there exists $n \in \mathbb{N}_0$ such that $|A| = |\{1, \ldots, n\}|$; otherwise A is **infinite**.

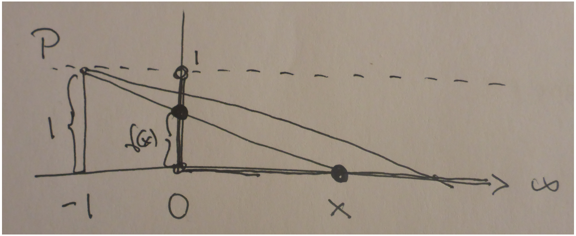
$\mathbb{R}$ is as big as the open interval $(0, 1)$

Theorem

$|\mathbb{R}| = |(0, 1)|$

Proof.

- $f : \mathbb{R}^+ \to (0, 1), \ x \mapsto \frac{x}{x+1}$, is bijective.
  This projects a point x on the positive x-axis to a point $f(x)$ between 0 and 1 on the y-axis:

- $g : \mathbb{R} \to \mathbb{R}^+, \ x \mapsto e^x$, is bijective.
- $f \circ g : \mathbb{R} \to (0, 1)$ is bijective.

Theorem

$|[0,1]| = |(0,1)|$

Proof.

HW $\square$

There are more reals than integers

Theorem (Cantor 1891)

$|\mathbb{N}| \neq |\mathbb{R}|$

Proof (Cantor's diagonal argument).

Show that no function $f : \mathbb{N} \to \mathbb{R}$ can be surjective. Consider

n	$f(n)$
1	$*.a_1 a_2 a_3 \ldots$
2	$*.b_1 b_2 b_3 \ldots$
3	$*.c_1 c_2 c_3 \ldots$
$\vdots$	

$a_i, b_i, \ldots$ digits in decimal expansion

Let $z \in \mathbb{R}$ such that the n-th decimal place of z is distinct from the n-the decimal place of $f(n)$ for all $n \in \mathbb{N}$:

$$z = 0.z_1 z_2 z_2 \ldots \text{ with } z_1 \neq a_1,\ z_2 \neq b_2,\ z_3 \neq b_3, \ldots$$

Then $z \neq f(n)$ for all $n \in \mathbb{N}$. Hence f is not surjective. $\square$

There are different sizes of infinite sets!

Definition

A set A is **countably infinite** if $|A| = |\mathbb{N}|$. The cardinality of $\mathbb{N}$ is $\aleph_0 := |\mathbb{N}|$ ('aleph zero', from Hebrew alphabet).
A is **uncountable** if A is infinite and $|A| \neq |\mathbb{N}|$.

Note

Every infinite set A has a countably infinite subset,

$$\begin{array}{cccccc} 1, & 2, & \ldots & n, & n+1, & \ldots \in \mathbb{N} \\ \downarrow & \downarrow & & \downarrow & \downarrow & \\ a_1, & a_2, & \ldots & a_n, & a_{n+1}, & \ldots \in A \end{array}$$

$\aleph_0$ is the smallest size an infinite set can have (the first infinite cardinal).

Example

$\mathbb{N}, \mathbb{Z}, \mathbb{N} \times \mathbb{N}, \mathbb{Q} \ldots$ are countably infinite.
$\mathbb{R}, [0, 1], \mathbb{C}, P(\mathbb{N}) \ldots$ are uncountable.

Why countable?

Note

A is countably infinite iff its elements can be enumerated as $a_1, a_2, a_3, \ldots$

Such an enumeration is just a bijection $\mathbb{N} \to A$, $\begin{array}{c} 1 \mapsto a_1 \\ 2 \mapsto a_2 \\ \vdots \end{array}$

Example

1. The set of prime numbers $p_1, p_2, \ldots$ can be enumerated, hence is countably infinite.
2. The elements of $\mathbb{R}$ cannot be enumerated one after the other by Cantor's diagonal argument.

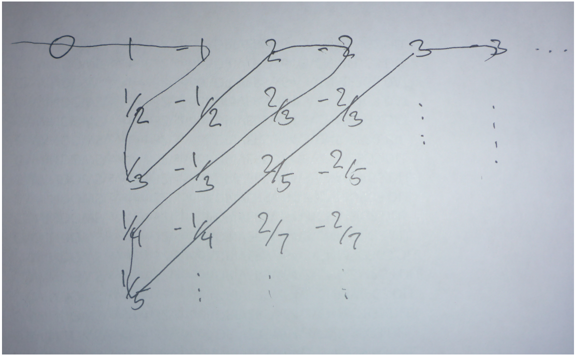
$\mathbb{Q}$ is countable

Theorem

$|\mathbb{Q}| = \aleph_0$

Proof.

Enumerate $\mathbb{Q}$



$\square$

Similarly $\mathbb{N} \times \mathbb{N}, \mathbb{Z}^3, \ldots$ can be enumerated.