Review 1: Sets

Peter Mayr

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Zermelo-Fraenkel Set Theory (ZF)

To avoid contradictions like Russell's paradox (the set of all sets that do not contain itself) one has to be careful how to define sets. The following axioms were proposed by Zermelo and Fraenkel:

1. **Axiom of extensionality.** Sets A and B are equal if they have the same elements.
2. **Axiom of regularity.** Every non-empty set A has an element B such that A and B are disjoint. [a]
3. **Axiom of specification.** For a set A and a property P,

   $$\{x \in A \ : \ x \text{ satisfies } P\}$$

   is a set.
4. **Axiom of pairing.** For any two sets A and B, there exists the set $\{A, B\}$.

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[a] With the Axiom of pairing, this implies that no set A is an element of itself; hence there is no set of all sets.

5. **Axiom of unions.** For a set I and sets A_i for $i \in I$,

$$\bigcup_{i \in I} A_i := \{x \ : \ x \in A_i \text{ for some } i \in I\}$$

   is a set.

6. **Axiom of replacement.** For a set A and a function f,

$$f(A) := \{f(x) \ : \ x \in A\}$$

   is a set.

7. **Axiom of infinity.** There exists a set that has infinitely many elements (e.g. $\mathbb{N}$).

8. **Axiom of power set.** For every set A, there exists the set of all its subsets

$$P(A) := \{B \ : \ B \subseteq A\}.$$

Question

Why is there no axiom of intersection?

$A \cap B = \{x \in A \ : \ x \in B\}$ exists by Axiom of specification.

Zermelo-Fraenkel with Choice (ZFC)

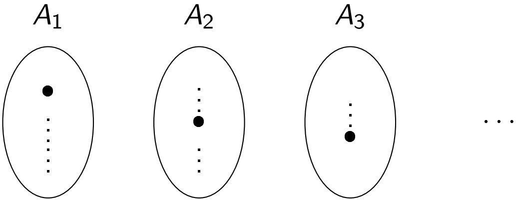
Some results in Math require the following additional axiom:

9. **Axiom of choice.** For a set I and non-empty sets A_i for $i \in I$, the Cartesian product $\prod_{i \in I} A_i$ is non-empty.

Follows from ZF for finite I.

Equivalent formulations of the Axiom of Choice

- For a set I and non-empty sets A_i for $i \in I$, there exists a function $f \colon I \to \bigcup_{i \in I} A_i$ such that $f(i) \in A_i$.
  [Non-constructive f **chooses** an element $f(i)$ from every A_i.]



Equivalent formulations of the Axiom of Choice, continued

- **Well-ordering Theorem:** On every set A there exists a partial order $\leq$ that is
  - total (linear), i.e., $\forall x, y \in A \colon x \leq y$ or $y \leq x$, and
  - well-ordered, i.e., every non-empty subset of A has a unique smallest element with respect to $\leq$.

Example

- $\mathbb{N}$ is well-ordered by the usual $\leq$.
- $\mathbb{R}$ is not because e.g. $(0, 1)$ has no smallest element w.r.t $\leq$. Still if you accept AC, there must be some well-ordering on $\mathbb{R}$.

Oh my god!

- ZF (and ZFC) were proposed to avoid paradoxes and contradictions in Math.
- They are now widely accepted as foundations of all Math.
- However we are not sure that ZF is free of contradictions.
- More precisely, by Gödel's Second Incompleteness Theorem:

  If ZF does not yield any contradiction, then it is also not strong enough to prove that it does not yield a contradiction.
- Do you want to know more? Take a class like 'Math 4000 – Foundations of Math' this Spring.

The Inclusion-Exclusion Principle

Theorem (Inclusion-Exclusion Principle)

Let $n \in \mathbb{N}$ and $A_1, \ldots, A_n$ be finite sets. Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \ldots$$

$$+(-1)^{k-1} \sum_{1 \le i_1 < \ldots i_k \le n} |A_{i_1} \cap \ldots A_{i_k}| + \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|$$

$$= \sum_{\emptyset \neq S \subseteq \{1,\ldots,n\}} (-1)^{|S|-1} \left|\bigcap_{i \in S} A_i\right|.$$

Proof by induction on n

- Base case $n = 1$: clear
- Base case $n = 2$: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$ since $|A_1| + |A_2|$ counts $|A_1 \cap A_2|$ twice.

- Induction assumption for fixed n:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset\neq S\subseteq\{1,\dots,n\}} (-1)^{|S|-1} \left|\bigcap_{i\in S} A_i\right|$$

- Induction step:

$$\left|\bigcup_{i=1}^{n+1} A_i\right| = |\underbrace{\bigcup_{i=1}^{n} A_i}_{=B} \cup A_{n+1}|$$

$$= |B| + |A_{n+1}| - |B \cap A_{n+1}| \qquad \text{by base case for 2}$$

$$= |B|+|A_{n+1}|-\left|\bigcup_{i=1}^{n}(A_i \cap A_{n+1})\right| \qquad \text{by distributivity of } \cap, \cup$$

Using the induction assumption for $|B|$ and for $|\bigcup_{i=1}^{n}(A_i \cap A_{n+1})|$

$$= \sum_{\emptyset\neq S\subseteq\{1,\dots,n\}} (-1)^{|S|-1} \left|\bigcap_{i\in S} A_i\right| + |A_{n+1}| - \sum_{\emptyset\neq S\subseteq\{1,\dots,n\}} (-1)^{|S|-1} \left|\bigcap_{i\in S}(A_i \cap A_{n+1})\right|$$

$$\left|\bigcup_{i=1}^{n+1} A_i\right|$$

$$= \sum_{\emptyset \neq S \subseteq \{1,\ldots,n\}} (-1)^{|S|-1} \left|\bigcap_{i\in S} A_i\right| + |A_{n+1}| - \sum_{\emptyset \neq S \subseteq \{1,\ldots,n\}} (-1)^{|S|-1} \left|\bigcap_{i\in S} (A_i \cap A_{n+1})\right|$$

$$= \sum_{\emptyset \neq S \subseteq \{1,\ldots,n\}} (-1)^{|S|-1} \left|\bigcap_{i\in S} A_i\right| + |A_{n+1}| - \sum_{\{n+1\} \subsetneq T \subseteq \{1,\ldots,n+1\}} (-1)^{|T|-2} \left|\bigcap_{i\in T} A_i\right|$$

$$= \underbrace{\sum_{\emptyset \neq S \subseteq \{1,\ldots,n\}} (-1)^{|S|-1} \left|\bigcap_{i\in S} A_i\right|}_{\text{all intersections without } A_{n+1}} + \underbrace{|A_{n+1}| + \sum_{\{n+1\} \subsetneq T \subseteq \{1,\ldots,n+1\}} (-1)^{|T|-1} \left|\bigcap_{i\in T} A_i\right|}_{\text{all intersections with } A_{n+1}}$$

$$= \sum_{\emptyset \neq S \subseteq \{1,\ldots,n+1\}} (-1)^{|S|-1} \left|\bigcap_{i\in S} A_i\right| \qquad \square$$