Relations 1

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Life (Math) is full of statements how 2 things relate to each other

Example

$x = 3$

$-2 \leq 5$

$3 \mid 12$

$1 \in \mathbb{N}$

$x \mapsto x^2$

$5 \equiv 1 \pmod{2}$

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R is represented by those pairs that go together:

$$R = \{(a, c), (a, s), (b, f), (b, s)\} \subseteq \{a, b\} \times \{c, f, s\}$$

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3. $T := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \ : \ n | x - y\}$ for fixed $n \in \mathbb{N}$
   $T \ = \equiv_n$, the **congruence modulo** n
   e.g. $0 \equiv_n n \equiv_n 2n \ldots$

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More about functions later.

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Beware: antisymmetric is not the same as 'not symmetric'.

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  By counterexample. Give explicit $x, y \in \mathbb{Z}$ such that xRy but not yRx.

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  Example, $0 \leq 1$ but $1 \not\leq 0$

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$=$ is reflexive, symmetric, antisymmetric, transitive, a function.

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