COMMENTS ON WRITING PROJECT 'FUNCTIONS'

PETER MAYR

The following issues came up repeatedly in the previous assignment of counting functions.

(1) Before submission, always proofread what you wrote. Check for typos, logical inconsistencies, anything that the reader may stumble over.

(2) Always write in complete sentences. Something like 'Injective: $\forall x, y \in A : f(x) = f(y) \Rightarrow x = y$' or 'All functions: n^k' is not selfcontained and does not make sense.

(3) Use the definition environment to explain technical terms like 'injective,...' before their first use.

(4) In every definition, theorem, ... specify and quantify all variables $k, n, ...$ that you use.

(5) Be careful to state under which conditions your results hold. Don't forget about any special cases.

Also the special cases like 'there are no surjective functions from $\{1, \ldots, k\}$ to $\{1, \ldots, n\}$ if $k < n$' require a proof.

(6) Use the definitions you learned in class in your proofs, not some selfmade approximations.

A rather frequent mistake is to say 'a function $f\colon A \to B$ is injective if it maps every $x \in A$ to a unique element $f(x) \in B$'. This means that x is mapped to exactly one value $f(x)$. But this is just the defining property of a function: 'it maps every x in the domain to exactly one y in the codomain.' This is not injectivity!

If you want to describe injectivity colloquially, you can say: f is injective if it maps any pair of distinct elements to distinct elements.

One example how you can organize your main theorem efficiently:

Theorem. *Let $k, n \in \mathbb{N}$ and write $[k] := \{1, \ldots, k\}, [n] := \{1, \ldots, n\}$.*

(1) *There are n^k functions from $[k]$ to $[n]$.*

(2) ...

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(3) *If $k \geq n$, there are $\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^k$ surjective functions from $[k]$ to $[n]$; if $k < n$, there are none.*
(4) ...

Proof. ...

(3) If $k < n$, no function $f\colon [k] \to [n]$ is surjective because ...

Assume $k \geq n$. To obtain the number of surjective functions $[k] \to [n]$ we take the number of all functions n^k and subtract the number of non-surjective functions, that is, those functions whose range is a proper subset of $[n]$. Now $[n]$ has $\binom{n}{n-1}$ different subsets with $n-1$ elements. By (1) there are $(n-1)^k$ functions from $[k]$ into any of these subsets, hence $\binom{n}{n-1}(n-1)^k$ non-surjective functions in total. However this overcounts the functions into any of the $\binom{n}{n-2}$ subsets with $n-2$ elements, which we have to add again. Then we need to subtract the functions into subsets with $n-3$ elements, and so forth. We use the Inclusion-Exclusion Principle to obtain that the number of surjections from $[k]$ to $[n]$ as

$$n^k - \binom{n}{n-1}(n-1)^k + \binom{n}{n-2}(n-2)^k - \cdots \pm \binom{n}{1}1^k = \sum_{i=1}^{n} \ldots$$

... □