Math 2001 - Assignment 13

Due December 4, 2020

(1) Let $f\colon A \to B, g\colon B \to C$. Show that
  (a) If $g \circ f$ is injective, then f is injective.
  (b) If $g \circ f$ is surjective, then g is surjective.
  Hint: Use contrapositive proofs.
  Give examples for f, g on $A = B = C = \mathbb{N}$ such that
  (c) $g \circ f$ is injective but g is not injective;
  (d) $g \circ f$ is surjective but f is not surjective.

Proof.

(a) Assume f is not injective, that is, we have $x, y \in A$ such that $x \neq y$ but $f(x) = f(y)$. Then $g(f(x)) = g(f(y))$ as well. Hence $g \circ f$ is not injective.

(b) Assume g is not surjective, that is, $g(B) \neq C$. Since $g(f(A)) \subseteq g(B)$, $g(f(A))$ cannot be all of C either. Hence $g \circ f$ is not surjective.

(c) If $g \circ f$ is injective, then g restricted to $f(A)$ has to be injective. But it does not matter what g does on $B - f(A)$. E.g., let $f\colon \mathbb{N} \to \mathbb{N},\ x \mapsto 2x,\ g\colon \mathbb{N} \to \mathbb{N},\ x \mapsto \lceil \frac{x}{2} \rceil$ where $\lceil r \rceil$ is the smallest integer z such that $z \geq r$. Then $g \circ f = \mathrm{id}_{\mathbb{N}}$ is injective but g is not.

(d) If $g \circ f$ is surjective, then $g(f(A)) = C$ but it does mean that $f(A)$ needs to be all of B.
E.g. as in (c) $g \circ f = \mathrm{id}_{\mathbb{N}}$ is surjective but f is not.

(2) (a) Show that

$$f : \mathbb{R} - \{1\} \to \mathbb{R} - \{2\}, x \mapsto \frac{2x+1}{x-1}$$

is bijective.

(b) Determine f^{-1}.

Solution. (a) Injective: Let $x, y \in \mathbb{R} - \{1\}$ such that $f(x) = f(y)$. Show $x = y$. We have

$$\begin{aligned} \frac{2x+1}{x-1} &= \frac{2y+1}{y-1} \\ (2x+1)(y-1) &= (2y+1)(x-1) \\ \ldots &= \ldots \\ x &= y \end{aligned}$$

Hence f is injective.

Surjective: Let $y \in \mathbb{R} - \{2\}$ such that $f(x) = y$. Solve for $x \in \mathbb{R} - \{1\}$.

$$\begin{aligned} y &= \frac{2x+1}{x-1} \\ y(x-1) &= 2x+1 \\ -y-1 &= x(-y+2) \\ \frac{y+1}{y-2} &= x \end{aligned}$$

So we found $x \in \mathbb{R} - \{1\}$ such that $f(x) = y$ and hence f is surjective.

Thus f is bijective.

(b) From the proof of surjectivity, we see

$$f^{-1} \colon \mathbb{R} - \{2\} \to \mathbb{R} - \{1\}, \ y \mapsto \frac{y+1}{y-2}$$

Note. Checking surjectivity and finding the inverse is pretty much the same work. So you may just try to find f^{-1} straight away without bothering about injectivity and surjectivity first. If $f(x) = y$ does not have a unique solution, then you'll see a failure of injectivity or surjectivity anyway.

(3) Try to you find an inverse for $f \colon \mathbb{R} \to \mathbb{R}^+, \ x \mapsto e^{x^3+1}$. Is f bijective?

Solution: Given $y \in \mathbb{R}^+$, find $x \in \mathbb{R}$ such that $f(x) = y$. So we solve

$$\begin{aligned} e^{x^3+1} &= y \\ x^3 + 1 &= \log y \\ x &= (\log y - 1)^{\frac{1}{3}} \end{aligned}$$

So

$$f^{-1} \colon \mathbb{R}^+ \to \mathbb{R}, y \mapsto (\log y - 1)^{\frac{1}{3}}$$

By checking $f \circ f^{-1} = \mathrm{id}_{\mathbb{R}^+}$ and $f^{-1} \circ f = \mathrm{id}_{\mathbb{R}}$ we see that f is bijective.

(4) Find the inverse for $f \colon \mathbb{R}^2 \to \mathbb{R}^2, \ (x, y) \mapsto (3x + y, x - 2y)$.

Solution: Given $(u, v) \in \mathbb{R}^2$, find $(x, y) \in \mathbb{R}^2$ such that $f(x, y) = (u, v)$. So we solve

$$\begin{aligned} 3x + y &= u \\ x - 2y &= v \end{aligned}$$

Multiplying the second equation by 3 and subtracting from the first yields

$$7y = u - 3v$$

So $y = \frac{u-3v}{7}$. Inserting in the first equation yields $x = \frac{2u+v}{7}$. Hence

$$f^{-1} \colon \mathbb{R}^2 \to \mathbb{R}^2,\ (u, v) \mapsto (\frac{2u+v}{7}, \frac{u-3v}{7})$$

(5) Let U be a set, and let c be the function on the power set of U that maps every set to its complement in U, i.e.,

$$c \colon P(U) \to P(U), X \mapsto \bar{X}.$$

Determine c^{-1} if it exists.

Solution: Given $Y \in P(U)$, find $X \in P(U)$ such that $\bar{X} = Y$. Take the complement again to get $X = \bar{\bar{X}} = \bar{Y}$.

Hence $c = c^{-1}$ is its own inverse. This can also be seen by $c \circ c = \mathrm{id}_{P(U)}$.

(6) Give an explicit bijection $f \colon [0, 1] \to (0, 1)$. Show that your function f is bijective.

Hint: Consider some sequence $0, 1, \ldots$ in $[0, 1]$ and use the idea of Hilbert's hotel.

Solution: We need f to push the endpoints $0, 1$ into the open interval $(0, 1)$. But then $f(0), f(1)$ have to be mapped be pushed somewhere else again and so forth. This works like for a bijection $\mathbb{N} \to \mathbb{N} - \{1, 2\}$ (Hilbert's hotel).

Fix a (countable) sequence in $(0, 1)$, say $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$, push each element 2 steps down and map $0, 1$ to the first 2 elements that are now free. The remaining elements of $(0, 1)$ can stay at their place. Formally define

$$f \colon [0, 1] \to (0, 1),\ x \mapsto \begin{cases} 1/2 & \text{if } x = 0, \\ \frac{1}{n+2} & \text{if } x = \frac{1}{n} \text{ for } n \in \mathbb{N}, \\ x & \text{else.} \end{cases}$$

From the definition by cases its clear that f is injective and surjective.