Math 2001 - Assignment 12

Due November 20, 2020

(1) Complete the proof of the following:

Theorem. Let $\{A_i \ : \ i \in I\}$ be a partition of a set A. Then

$$x \sim y \text{ if } \exists i \in I : x, y \in A_i$$

defines an equivalence relation on A with equivalence classes A_i for $i \in I$.

Proof: For reflexivity: Let $x \in A$. Since $A = \bigcup_{i \in I} A_i$ by the definition of a partition, we have $i \in I$ such that $x \in A_i$. Hence $x \sim x$.

For symmetry: Let $x, y \in A$. Assume $x \sim y$, that is, $x, y \in A_i$ for some $i \in I$. Then $y, x \in A_i$ and $y \sim x$.

For transitivity: Let $x, y, z \in A$. Assume $x \sim y$ and $y \sim z$. Then we have $i \in I$ such that $x, y \in A_i$ and $j \in I$ such that $y, z \in A_j$. Since $y \in A_i \cap A_j$, we have $i = j$ by the definition of a partition. Hence $x, z \in A_i$ and $x \sim z$.

This completes the proof that $\sim$ is an equivalence relation.

Finally for every $x \in A$, the class $[x]_\sim = A_i$ for the unique $i \in I$ such that $x \in A_i$. □

(2) (a) Given finite sets A and B. How many different relations are there from A to B?

Solution: Since relations are just subsets of $A \times B$, there are $|P(A \times B)| = 2^{|A||B|}$ many.

(b) How many different equivalence relations are there on $A = \{1, 2, 3\}$? Describe them all by listing the partitions of A.

Solution: There are 5 ways to partition A into equivalence classes:
$\{1\}, \{2\}, \{3\}$ (these are the classes of $=$)
$\{1\}, \{2, 3\}$
$\{1, 3\}, \{2\}$
$\{1, 2\}, \{3\}$
$\{1, 2, 3\}$
Hence there are 5 partitions/equivalence relations.

(3) (a) Give the tables for addition and multiplication for $\mathbb{Z}_6$.

Solution:

$+$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
$\vdots$						

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
$\vdots$						

(b) Dividing by $[a]$ in $\mathbb{Z}_n$ means solving an equation $[a] \cdot [x] = [1]$ for $[x]$. Solve $[8] \cdot [x] = [1]$ in $\mathbb{Z}_{37}$.
Hint: Use the Euclidean algorithm to solve $8x \equiv 1 \mod 37$.

Solution: Solve $8x + 37y = 1$ by the Euclidean algorithm to get the Bezout coefficient $x = 14$.

(4) (a) Give domain, codomain, and range of $f : \mathbb{Z} \to \mathbb{N},\ x \mapsto x^2 + 1$. What is $f(3)$?
(b) Is f one-to-one, onto, bijective?
(c) Determine $f(\{2x \ : \ x \in \mathbb{Z}\})$ and $f^{-1}(\{1, 2, 3, \dots, 10\})$.

Solution.
(a) domain $\mathbb{Z}$, codomain $\mathbb{N}$, range $\{x^2 + 1 \ : \ x \in \mathbb{Z}\}$, $f(3) = 10$
(b) not injective since e.g. $f(1) = f(-1)$,
not surjective since e.g. $\nexists x \in \mathbb{Z} : f(x) = 3$,
hence not bijective
(c) $f(\{2x \ : \ x \in \mathbb{Z}\}) = \{4x^2 + 1 \ : \ x \in \mathbb{Z}\}$,
$f^{-1}(\{1, 2, 3, \dots, 10\}) = \{-3, -2, -1, 0, 1, 2, 3\}$

(5) Give examples for
(a) a function $f : \mathbb{N} \to \mathbb{N}$ that is not injective but surjective;
(b) a function $g : \{1, 2, 3\} \to \{1, 2\}$ that is neither injective nor surjective;
(c) a bijective function $h : \{1, 2, 3\} \to \{1, 2\}$.

Solution.
(a) E.g. $f(x) = \lceil \frac{x}{2} \rceil$, the smallest integer greater or equal to $\frac{x}{2}$
(b) any constant function
(c) Not possible: Because the codomain is smaller than the domain, there is no injective h.

(6) Let A, B be finite sets with $|A| = |B|$, and let $f\colon A \to B$. Show that f is injective iff f is surjective.

Is this true for functions between infinite sets as well? Prove it or give counterexamples for each direction.

Proof. injective $\Rightarrow$ surjective: Assume f is injective. Then $|A| = |f(A)|$. Since $f(A) \subseteq B$ and $|A| = |B|$, we get that $f(A) = B$. Hence f is surjective.

surjective $\Rightarrow$ injective: Assume $f(A) = B$. Then $|f(A)| = |B| = |A|$ yields that f is injective. If $x \neq y$ in A, then $f(x) \neq f(y)$ because otherwise $|f(A)| < |A|$. $\square$

For infinite sets, e.g. $f\colon \mathbb{N} \to \mathbb{N}$, $x \mapsto x + 1$ is injective but not surjective; exercise (5a) gives a surjective function that is not injective.