Math 2001 - Assignment 12

Due November 20, 2020

(1) Complete the proof of the following:

Theorem. Let $\{A_i \ : \ i \in I\}$ be a partition of a set A. Then

$$x \sim y \text{ if } \exists i \in I : x, y \in A_i$$

defines an equivalence relation on A with equivalence classes A_i for $i \in I$.

Proof: For reflexivity: Let $x \in A$. Since $A =$______________ by the definition of ______________, we have $i \in I$ such that $x \in$_____. Hence $x \sim$_____.

For ______________: Let $x, y \in A$. Assume $x \sim y$, that is, ____________ for some $i \in I$. Then $y, x \in A_i$ and __________.

For transitivity: Let __________. Assume $x \sim y$ and $y \sim z$. Then we have $i \in I$ such that ______________ and $j \in I$ such that ______________. Since _____$\in A_i \cap A_j$, we have __________ by the definition of a partition. Hence ____________ and $x \sim z$.

This completes the proof that $\sim$ is ______________.

Finally for every $x \in A$, the class $[x]_\sim =$ __________ for the unique $i \in I$ such that $x \in$ __________. □

(2) (a) Given finite sets A and B. How many different relations are there from A to B?

(b) How many different equivalence relations are there on $A = \{1, 2, 3\}$? Describe them all by listing the different partitions of A.

(3) (a) Give the tables for addition and multiplication for $\mathbb{Z}_6$.

(b) Dividing by $[a]$ in $\mathbb{Z}_n$ means solving an equation $[a] \cdot [x] = [1]$ for $[x]$. Solve $[8] \cdot [x] = [1]$ in $\mathbb{Z}_{37}$.
Hint: Use the Euclidean algorithm to solve $8x \equiv 1 \mod 37$.

(4) (a) Give domain, codomain, and range of $f \colon \mathbb{Z} \to \mathbb{N},\ x \mapsto x^2 + 1$. What is $f(3)$?

(b) Is f one-to-one, onto, bijective?

(c) Determine $f(\{2x \ : \ x \in \mathbb{Z}\})$ and $f^{-1}(\{1, 2, 3, \ldots, 10\})$.

(5) Give examples for

(a) a function $f \colon \mathbb{N} \to \mathbb{N}$ that is not injective but surjective;

(b) a function $g \colon \{1, 2, 3\} \to \{1, 2\}$ that is neither injective nor surjective;

(c) a bijective function $h \colon \{1, 2, 3\} \to \{1, 2\}$.

(6) Let A, B be finite sets with $|A| = |B|$, and let $f \colon A \to B$. Show that f is injective iff f is surjective.

Is this true for functions between infinite sets $A = B = \mathbb{N}$ as well? Prove it or give counterexamples for each direction.