Math 2001 - Assignment 11

Due November 13, 2020

(1) Define a sequence of integers $a_1 := 1, a_2 := 1$ and

$$a_n := 2a_{n-1} + a_{n-2} \text{ for } n \geq 3.$$

Prove that a_n is odd for all $n \in \mathbb{N}$ by strong induction.

Proof by strong induction on n: Induction basis for $n = 1, 2$: holds by definition of a_1, a_2.

Strong induction assumption: Assume a_i is odd for all $i \leq k$.

Induction step: Show that a_{k+1} is odd.

By definition $a_{k+1} = 2a_k + a_{k-1}$. Since a_{k-1} is odd by the strong induction assumption, a_{k+1} is the sum of an even and an odd integer. Hence a_{k+1} is odd. □

(2) Prove or disprove that the following relations are reflexive, symmetric, antisymmetric, transitive. Which are equivalences, which partial orders?

(a) $\neq$ on $\mathbb{Z}$

Solution.

- not reflexive since e.g. it is not true that $0 \neq 0$
- symmetric since $\forall x, y \in \mathbb{Z}$: $x \neq y \Rightarrow y \neq x$
- not antisymmetric since e.g. $0 \neq 1$ and $1 \neq 0$ but it is not true that $0 = 1$
- not transitive since e.g. $0 \neq 1$ and $1 \neq 0$ but it is not true that $0 \neq 0$
- Hence $\neq$ is neither an equivalence nor a partial order.

(b) $\subseteq$ on the power set $P(A)$ of a set A

Solution.

- reflexive since $X \subseteq X$ for every set $X \in P(A)$.
- not symmetric if $A \neq \emptyset$. Then $\emptyset \subseteq A$ but $A \not\subseteq \emptyset$.
- antisymmetric since $X \subseteq Y$ and $Y \subseteq X$ implies $X = Y$ for all $X, Y \in P(A)$.
- transitive since $X \subseteq Y$ and $Y \subseteq Z$ implies $X \subseteq Z$ for all $X, Y, Z \in P(A)$.
- Hence $\subseteq$ is not an equivalence but a partial order.

(3) Prove or disprove that the following relations are reflexive, symmetric, antisymmetric, transitive. Which are equivalences, which partial orders?

(a) $|$ (divides) on $\mathbb{N}$
Solution.
- reflexive since $x|x$ for every $x \in \mathbb{N}$.
- not symmetric since e.g. $1|2$ but $2 \not| 1$.
- antisymmetric since $x|y$ and $y|x$ implies $x = y$ for all $x, y \in \mathbb{N}$.
- transitive since $x|y$ and $y|z$ implies $x|z$ for all $x, y, z \in \mathbb{N}$.
- Hence $|$ is not an equivalence but a partial order.

(b) $R = \{(x, y) \in \mathbb{R} \colon |x - y| \leq 1\}$
Solution.
- reflexive since $|x - x| = 0 \leq 1$ for every $x \in \mathbb{R}$.
- symmetric since $|x - y| = |y - x|$ for all $x, y \in \mathbb{R}$.
- not antisymmetric since e.g. $|0-1| \leq 1$ and $|1-0| \leq 1$ but $0 \neq 1$
- not transitive since e.g. $|0 - 1| \leq 1$ and $|1 - 2| \leq 1$ but $|0 - 2| > 1$
- Hence R is neither an equivalence nor a partial order.

(4) List the equivalence classes for these equivalence relations:
(a) The relation $\sim$ on subsets A, B of $\{1, 2, 3\}$ where $A \sim B$ if $|A| = |B|$.
Solution.
$[\emptyset] = \{\emptyset\}$ . . . the class of sets of size 0
$[\{1\}] = \{\{1\}, \{2\}, \{3\}\}$ . . . the class of sets of size 1
$[\{1, 2\}] = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$ .. class of sets of size 2
$[\{1, 2, 3\}] = \{\{1, 2, 3\}\}$ . . . class of sets of size 3
(b) $R = \{(x, y) \in \mathbb{Z} \colon |x| = |y|\}$ on $\mathbb{Z}$
Solution. $[x] = \{-x, x\}$ for $x \in \mathbb{Z}$

(5) (a) Given finite sets A and B. How many different relations are there from A to B?
(b) How many different equivalence relations are there on $A = \{1, 2, 3\}$? Describe them all by listing the partitions of A.
Solution: Postponed for next week.

(6) Let $\sim$ be an equivalence relation on a set A, let $a, b \in A$. Let $[a]$ denote the equivalence class of a modulo $\sim$. Show that

$$a \not\sim b \text{ iff } [a] \cap [b] = \emptyset.$$

Solution: $\Rightarrow$ by contrapositive proof: Assume $[a] \cap [b] \neq \emptyset$, that is, we have $c \in [a] \cap [b]$. Then $c \sim a$ and $c \sim b$. By symmetry we get $a \sim c$ and by transitivity $a \sim b$.

$\Leftarrow$: Assume $[a] \cap [b] = \emptyset$. Since $a \in [a]$, we then get $a \notin [b] = \{x \in A : x \sim b\}$. Hence $a \not\sim b$. □