Math 2001 - Assignment 10

Due November 6, 2020

The first 4 problems are meant to be revision for the midterm. Do them before Wednesday.

(1) (a) How many permutations of the alphabet $a, \dots, z$ contain the word "fish"?

Solution: Consider "fish" as one item together with the remaining 22 letters to get 23! permutations.

(b) How many permutations of the alphabet do not contain any of the words "fish", "rat" or "bird"?

Solution:

permutations containing	fish	23!
	rat	24!
	bird	23!
	fish and rat	21!
	fish and bird	0
	bird and rat	0

So by inclusion-exclusion: $26! - 23! - 24! - 23! + 21!$ permutations without "fish", "rat" or "bird".

(2) A regular poker card set has 4 suits and 13 cards for each suit.

(a) How many sets of 5 cards (out of 52) are there with 4 cards of one kind?

Solution: $13 * 48$

(b) How many sets of 5 cards (out of 52) are there with all cards of the same suit?

Solution: $4 * \binom{13}{5}$

(3) [1, Chapter 10, exercise 8] Show that for every $n \in \mathbb{N}$:

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

Proof by induction on n:

Inductive base: For n=1, it is true that $\frac{1}{2!} = 1 - \frac{1}{2!}$.

Induction assumption: For a fixed n we have

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

Inductive step: Show

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{n+1}{(n+2)!}=1-\frac{1}{(n+2)!}$$

Just start with the left hand side and simplify it using the induction assumption:

$$\begin{aligned}\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{n+1}{(n+2)!} &= 1-\frac{1}{(n+1)!}+\frac{n+1}{(n+2)!} \text{ by induction assumption}\\ &= 1-\frac{n+2}{(n+2)!}+\frac{n+1}{(n+2)!}\\ &= 1-\frac{1}{(n+2)!}\end{aligned}$$

Thus the statement is true for all $n \in \mathbb{N}$. □

(4) Show by induction that for every natural number $n \geq 4$:

$$2^n \geq n^2$$

Proof by induction on n:

Induction basis for $n = 4$: $2^4 \geq 4^2$ holds.

Induction assumption: Assume $2^k \geq k^2$ holds for a particular $k \geq 4$.

Induction step: Show $2^{k+1} \geq k+1^2$.

Note

$$2^{k+1} = 2 \cdot 2^k \geq 2k^2 \text{ by induction assumption}$$

So we want to still show that

$$2k^2 \geq (k+1)^2. \tag{1}$$

To this end, look at the difference of both sides

$$\begin{aligned}2k^2-(k+1)^2 &= k^2-2k-1\\ &= (k-1)^2-2\\ &\geq 3^2-2 \text{ because } k \geq 4.\end{aligned}$$

In particular $2k^2-(k+1)^2 \geq 0$ which proves (1) and the induction step. □

(5) Let $p_1, p_2, \ldots$ denote the list of all primes. Show that for integers $a = \Pi_{i\in\mathbb{N}}\, p_i^{e_i}, b = \Pi_{i\in\mathbb{N}}\, p_i^{f_i}$ with $e_i, f_i \in \mathbb{N}_0$ for $i \in \mathbb{N}$,

$$\operatorname{lcm}(a,b) = \Pi_{i\in\mathbb{N}}\, p_i^{\max(e_i,f_i)}.$$

Proof:
a) First note that $\Pi_{i\in\mathbb{N}}\ p_i^{\max(e_i,f_i)}$ is an integer multiple of a since

$$\Pi_{i\in\mathbb{N}}\ p_i^{\max(e_i,f_i)} = a \cdot \underbrace{\Pi_{i\in\mathbb{N}}\ p_i^{\max(e_i,f_i)-e_i}}_{\in\mathbb{Z}}.$$

Similar $\Pi_{i\in\mathbb{N}}\ p_i^{\max(e_i,f_i)}$ is an integer multiple of b.
b) Next let $m = \Pi_{i\in\mathbb{N}}\ p_i^{g_i)}$ for $g_i \in \mathbb{N}_0$ a common multiple of a and b. Let $i \in \mathbb{N}$. Then $p_i^{e_i}$ divides m and by the Fundamental Theorem of Arithmetic, $p_i^{e_i}$ divides $p_i^{g_i}$. Hence $e_i \leq g_i$. Similarly $f_i \leq g_i$. Together they imply that $\max(e_i, f_i) \leq g_i$.

Thus for any common multiple m of a and b we have $\Pi_{i\in\mathbb{N}}\ p_i^{\max(e_i,f_i)} \leq m$ and the former is $\operatorname{lcm}(a,b)$. □

(6) Show for all $a, b \in \mathbb{N}$:

$$\gcd(a,b) \cdot \operatorname{lcm}(a,b) = ab$$

Hint: Use the formula for gcd and lcm from class and the previous problem.

Proof: Let $a = \prod_{i\in\mathbb{N}} p_i^{e_i}, b = \prod_{i\in\mathbb{N}} p_i^{f_i}$. By a lemma from class

$$\gcd(a,b) = \prod_{i\in\mathbb{N}} p_i^{\min(e_i,f_i)} \qquad \operatorname{lcm}(a,b) = \prod_{i\in\mathbb{N}} p_i^{\max(e_i,f_i)}$$

Now $ab = \prod_{i\in\mathbb{N}} p_i^{e_i+f_i}$ and

$$\gcd(a,b)\operatorname{lcm}(a,b) = \prod_{i\in\mathbb{N}} p_i^{\min(e_i,f_i)+\max(e_i,f_i)}.$$

Both numbers on the right hand side are equal since for any $e, f \in \mathbb{N}_0$

$$e + f = \min(e,f) + \max(e,f).$$

The proof of that is by case distinction:
Case 1, $e \leq f$: Then $\min(e,f) = e, \max(e,f) = f$ and $\min(e,f) + \max(e,f) = e + f$.
Case 2, $e > f$: Similar. □

References

[1] Richard Hammack. The Book of Proof. Creative Commons, 3rd edition, 2018. Available for free: `http://www.people.vcu.edu/~rhammack/BookOfProof/`