Math 2001 - Assignment 9

Due October 30, 2020

(1) [1, Chapter 6, exercise 8] Prove by contradiction: Let $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.

Proof by contradiction: Assume $a^2 + b^2 = c^2$ and a and b are odd. Then $a = 2x + 1$, $b = 2y + 1$ for some $x, y \in \mathbb{Z}$. So

$$c^2 = (2x+1)^2+(2y+1)^2 = 4x^2+4x+1+4y^2+4y+1 = 2(2x^2+2x+2y^2+2y+1).$$

In particular c^2 is even, which implies that c is even by Lemma 2 from class. So $c = 2z$ for some $z \in \mathbb{Z}$. The identity above yields

$$4z^2 = 2(2x^2 + 2x + 2y^2 + 2y + 1).$$

Further $2z^2 = 2(x^2 + x + y^2 + y) + 1$. Now $2z^2$ is even whereas $2(x^2 + x + y^2 + y) + 1$ is odd. This is a contradiction.

Hence our assumptions where false and the original statement is true. $\square$

(2) Prove for all $x, y \in \mathbb{R}$:
If x is rational and xy is irrational, then y is irrational.

Proof by contradiction: Seeking a contradiction suppose that x is rational and xy is irrational and y is rational.

Then $x = \frac{a}{b}$ and $y = \frac{c}{d}$ for $a, b, c, d \in \mathbb{Z}$. Hence $xy = \frac{ac}{bd}$ is rational contradicting the assumption that xy is irrational. $\square$

(3) Compute:
- (a) $3 \cdot 4 \mod 7$
- (b) $2 - 9 \mod 11$
- (c) $2^6 \mod 9$
- (d) Solve for $x \in \mathbb{Z}$: $13x \equiv 1 \mod 31$
  Hint: First solve the equation $13x + 31y = 1$ using the extended Euclidean algorithm.

Solution:
- (a) $3 \cdot 4 \mod 7 = 5$
- (b) $2 - 9 \mod 11 = 4$
- (c) $2^6 \mod 9 = 1$

(d) Determine $\gcd(13, 31)$ with Bezout coefficients:

$$\begin{array}{rrrl} & 31 & 13 & \\ 31 & 1 & 0 & \\ 13 & 0 & 1 & / \cdot (-2) \\ 5 & 1 & -2 & / \cdot (-2) \\ 3 & -2 & 5 & / \cdot (-1) \\ 2 & 3 & -7 & / \cdot (-1) \\ 1 & -5 & 12 & \end{array}$$

So $\gcd(31, 13) = 1 = (-5) \cdot 31 + 12 \cdot 13$.
Modulo 31 we get that $x = 12$ solves $13x \equiv 1 \mod 31$.

(4) Prove: Let $a, b, c, d \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $a \equiv b \mod n$ and $c \equiv d \mod n$, then $a + c \equiv b + d \mod n$.

Proof: Assume $a \equiv b \mod n$ and $c \equiv d \mod n$, that is, $n|a-b$ and $n|c-d$. By Lemma 1 of the handout on Integers we then have n divides their sum $(a-b)+(c-d)$. Hence $n|(a+c)-(b+d)$ and $a + c \equiv b + d \mod n$. □

(5) Prove by induction that for every $q \in \mathbb{R}$ with $q \neq 1$ and for every $n \in \mathbb{N}_0$:

$$1 + q^1 + q^2 + \cdots + q^n = \frac{1 - q^{n+1}}{1 - q}$$

Proof by induction on n:
Induction basis for $n = 0$: $q^0 = 1$ holds.

Induction assumption: Assume the formula holds for a particular $k \in \mathbb{N}$.

Induction step: Show the formula holds for $n = k + 1$.

$$\begin{aligned} \sum_{i=0}^{k+1} q^i &= \sum_{i=0}^{k} q^i + q^{k+1} \\ &= \frac{1 - q^{k+1}}{1 - q} + q^{k+1} \text{ by induction assumption} \\ &= \frac{1 - q^{k+1}}{1 - q} + \frac{q^{k+1} - q^{k+2}}{1 - q} \\ &= \frac{1 - q^{k+2}}{1 - q} \end{aligned}$$

Hence the induction step is proved and the formula holds for all $n \in \mathbb{N}$. □

(6) [1, Chapter 10, exercise 2] Show by induction that for every $n \in \mathbb{N}$:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof by induction on n:
Induction basis for $n = 1$: $1^2 = 1$ holds.

Induction assumption: Assume the formula holds for a particular $n \in \mathbb{N}$.

Induction step: Show the formula holds for $n + 1$.

$$\begin{aligned}
\sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \text{ by induction assumption} \\
&= \frac{(n+1)[n(2n+1) + 6(n+1)]}{6} \\
&= \frac{(n+1)[2n^2 + 7n + 6)]}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6}
\end{aligned}$$

Hence the induction step is proved and the formula holds for all $n \in \mathbb{N}$. □

References

[1] Richard Hammack. The Book of Proof. Creative Commons, 3rd edition, 2018. Available for free: `http://www.people.vcu.edu/~rhammack/BookOfProof/`