Math 2001 - Assignment 7

Due October 16, 2020

Be careful to write down every step in the proofs of problems 5,6 and reduce every statement to definitions or other statements that were already proved in class.

(1) (a) In how many ways can you put 7 identical balls into 3 rooms so that every room contains at least one ball?

Solution. First put 1 ball in each of the 3 rooms. Then 4 balls remain to be distributed in 3 rooms. This can be done in $\binom{4+2}{2}$ ways.

(b) [1, Section 3.8]: Exercise 14

Solution. How many permutations are there of the letters in the word "PEPPERMINT"?
We got 3 Ps, 2 Es, 1 of R,M,I,N,T each (10 letters total). So there are $\frac{10!}{3!\cdot 2!}$ different permutations.

(2) [1, Section 3.8]: Exercises 4 and 5

Solution. 4) A bag contains 20 identical red balls, 20 identical blue balls, 20 identical green balls, and 20 identical white balls. You reach in and grab 15 balls. How many different outcomes are possible?

You have 20 balls of each color, which is more than the 15 you need to pick. So this just means to divide 15 balls into 4 colors red, blue, green, white. This can be done in $\binom{15+3}{3}$ ways.

5) A bag contains 20 identical red balls, 20 identical blue balls, 20 identical green balls, and one white ball. You reach in and grab 15 balls. How many different outcomes are possible?

Case 1, omit the single white ball: Pick all 15 balls out of red, blue, green in $\binom{15+2}{2}$ ways.

Case 2, take the white ball: Pick the 14 remaining balls out of red, blue, green in $\binom{14+2}{2}$ ways.

So in total there are $\binom{17}{2} + \binom{16}{2}$ ways.

(3) How many 4-letter "words" can you form from the alphabet $A, \dots, Z$ if the letters are in alphabetical order and

(a) repetitions are not allowed, e.g., BEFS.
Hint: Such a word is uniquely determined by which letters from $A, \dots, Z$ occur in it. Which of our standard models for counting lists applies?

Solution. Same as the number of 4-subsets: $\binom{26}{4}$

(b) repetitions are allowed, e.g., BFFS.
Hint: Every such word is uniquely determined by how often each letter from $A, \dots, Z$ occurs in it.

Solution. Same as the number of 4-multisubsets: $\binom{4+25}{25}$.
Or same as nonnegative integer solutions for

$$x_1 + \cdots + x_{26} = 4,$$

or the number of ways of distributing 4 balls into 26 buckets: 4 balls are partitioned into 26 classes (many of them empty) by using 25 separators. The number of ways to do this is to choose 4 things out of 4+25: $\binom{4+26-1}{4}$

(4) (a) In how many different ways can you line up n people?

Solution. $n!$

(b) In how many different ways can you line up n people in a circle?

Hint: Since a circle has no beginning or end, two arrangements are the same if one is obtained from the other by rotation, e.g., the following are considered equal:

Represent arrangements by lists. When do 2 lists describe the same arrangement?

Solution. By a) we have $n!$ different permutations of n people on a line. If we bend the line to a circle, we see that

$$(a_1, \dots, a_n), (a_2, \dots, a_n, a_1), \dots, (a_n, a_1, \dots, a_{n-1})$$

all describe the same arrangement.
So we overcounted by a factor n. On a circle there are $\frac{n!}{n} = (n-1)!$ distinct arrangements.

(5) $a \in \mathbb{Z}$ is *even* if $a = 2n$ for some $n \in \mathbb{Z}$.
$a \in \mathbb{Z}$ is *odd* if $a = 2n + 1$ for some $n \in \mathbb{Z}$.

Prove:

(a) Let $a \in \mathbb{Z}$. If a is even, then a^2 is even.
Hint: Use a direct proof. Assume a is a multiple of 2. Show that a^2 is a multiple of 2.

Proof. Assume $a = 2q$ for some $q \in \mathbb{Z}$. Then $a^2 = 4q^2$ is a multiple of 2, hence even as well. $\square$

(b) If x is an odd integer, then 8 divides $x^2 - 1$.
Hint: Use a direct proof. Then split into 2 cases.

Proof. Assume $x = 2y+1$ for some $y \in \mathbb{Z}$. Then $x^2 - 1 = (2y+1)^2 - 1 = 4y^2 + 4y = 4y(y+1)$. We consider 2 cases:

(i) y is even: Then $y = 2z$ for some $z \in \mathbb{Z}$. Then $4y(y+1) = 8z(2z+1)$ is a multiple of 8.
(ii) y is odd: Then $y = 2z + 1$ for some $z \in \mathbb{Z}$. Then $4y(y + 1) = 4(2z + 1)(2z + 2) = 8(2z + 1)(z + 1)$ is a multiple of 8.

In both cases 8 divides $x^2 - 1$. $\square$

(6) Complete the proof from class that $\gcd(a, b) = \gcd(a-qb, b)$ for all $a, b, q \in Z$ with not both a and b equal 0.
Assume $d|a - qb$ and $d|b$. Show that $d|a$ and $d|b$.

Proof. Assume $d|a - qb$ and $d|b$. Then $a - qb = rd$ and $b = sd$ for some $r, s \in \mathbb{Z}$. We can compute a as

$$a = rd + qb = rd + qsd = (r + qs)d.$$

Hence $d|a$ and $d|b$. $\square$

References

[1] Richard Hammack. The Book of Proof. Creative Commons, 3rd edition, 2018. Available for free: `http://www.people.vcu.edu/~rhammack/BookOfProof/`