Math 2001 - Assignment 5

Due October 2, 2020

Problems 1-3 are review material for the first midterm on Sets and Logic September 30. So you should solve them before Wednesday!

(1) Simplify:
  (a) $\bigcup_{i=0}^{4} [i, 2i+1]$
  (b) $\bigcap_{n\in\mathbb{N}} \{x \in \mathbb{Z} : x \geq n\}$
  (c) $\bigcup_{x\in[0,1]} \{x\} \times [1,2]$
  (d) $\bigcup_{x\in[0,1]} \{x\} \times [0,x]$

Solution.
  (a) $\bigcup_{i=0}^{4} [i, 2i+1] = [0,1] \cup [1,3] \cup \cdots \cup [4,9] = [0,9]$
  (b) $\bigcap_{n\in\mathbb{N}} \{x \in \mathbb{Z} : x \geq n\} = \{1,2,3,\dots\} \cap \{2,3,4,\dots\} \cap \{3,4,\dots\} \cap \cdots = \emptyset$ since no integer x is greater than every natural number n
  (c) $\bigcup_{x\in[0,1]} \{x\} \times [1,2] = \{(x,y) : x \in [0,1], y \in [1,2]\} = [0,1] \times [1,2]$
  (d) $\bigcup_{x\in[0,1]} \{x\} \times [0,x] = \{(x,y) : x \in [0,1], y \in [0,x]\} = \{(x,y) : 0 \leq y \leq x \leq 1\}$

(2) (a) Is it true that for all statements P, Q, R:

$$(P \Rightarrow Q) \wedge P = Q$$

  Prove it or give a counter-example.
  (b) Show the distributive law $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$.

Solution. Consider truthtables:
  (a) $(P \Rightarrow Q) \wedge P = Q$ is not true for all P, Q, e.g., it fails for $P = F, Q = T$.
  (b) True by truthtable.

(3) Write using quantifiers and logical operations. Is the statement true? Give its negation.
  (a) The square of any real number is non-negative.
  (b) There exists an integer x such that $x^y = x$ for all integers y.
  (c) For all reals x and y we have that $xy = 0$ implies $x = 0$.

Solution.

(a) $\forall x \in \mathbb{R}\colon\ x^2 \geq 0$
True, since the product of 2 positive numbers is positive, the product of 2 negative numbers is positive, and $0 \cdot 0 = 0$.
Negation: $\exists x \in \mathbb{R}\colon\ x^2 < 0$
(b) $\exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z}\colon\ x^y = x$
True. Pick $x = 1$. Then we have $1^y = 1$ for all $y \in \mathbb{Z}$.
Negation: $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z}\colon\ x^y \neq x$
(c) $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\colon\ xy = 0 \Rightarrow x = 0$
False, e.g., for $x = 1, y = 0$ we have $xy = 0$ but not $x = 0$
Negation: $\exists x \in \mathbb{R}\ \exists y \in \mathbb{R}\colon\ xy = 0 \wedge x \neq 0$

(4) How many lists of length 4 are there with entries from A,..., Z if
(a) repetition is allowed,
(b) repetition is not allowed,
(c) repetition is not allowed and the list must contain A,
(d) repetition is allowed and the list must contain A.

Solution.
(a) 4-lists with repetition: 26^4
(b) 4-lists without repetition: $\frac{26!}{22!} = 358.800$
(c) Subtract all lists without A from (b): $\frac{26!}{22!} - \frac{25!}{21!}$
(d) Subtract all lists without A from (a): $26^4 - 25^4$

(5) [1, Section 3.3]: Exercise 2

Solution: 5 cards off a 52-card deck are lined up in a row. How many line-ups are there with all of the same suit?

First pick 1 out of 4 suits, then a list (not set) of 5 cards out of 13 of that particular suit

$$\binom{4}{1} \cdot \frac{13!}{8!} = 4 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9$$

(6) How many standard Colorado license plates (3 numbers followed by 3 letters) have at least one number or letter repeated?

Solution: number of all plates as computed in class: $10^3 26^3$

number of plates without repetition of any number or letter: $10 \cdot 9 \cdot 8 \cdot 26 \cdot 25 \cdot 24$

Subtract the above numbers to get the number of plates with one number or letter repeated: $10^3 26^3 - 10 \cdot 9 \cdot 8 \cdot 26 \cdot 25 \cdot 24 =$ 6.344.000

References

[1] Richard Hammack. The Book of Proof. Creative Commons, 3rd edition, 2018. Available for free: `http://www.people.vcu.edu/~rhammack/BookOfProof/`