Math 2001 - Assignment 3

Due September 18, 2020

(1) Simplify the following sets and justify your answers:

(a) $\bigcap_{n\in\mathbb{N}}\{nz \;:\; z \in \mathbb{Z}\}$ (b) $\bigcup_{x\in\mathbb{R}}[-x,x]$ (c) $\bigcap_{n\in\mathbb{N}}(-\frac{1}{n},\frac{1}{n})$

Solution.

(a) $\bigcap_{n\in\mathbb{N}}\{nz \;:\; z \in \mathbb{Z}\} = \{0\}$ because 0 is the only integer that is a multiple of every natural number.

(b) $\bigcup_{x\in\mathbb{R}}[-x,x] = \mathbb{R}$ because every real x is in some interval of the union, namely $[-|x|,|x|]$.

(c) $\bigcap_{n\in\mathbb{N}}(-\frac{1}{n},\frac{1}{n}) = \{0\}$ because 0 is the only real number that is contained in every interval $(-\frac{1}{n},\frac{1}{n})$ for $n \in \mathbb{N}$

(2) Are the following statements? If so, determine whether they are true or false and write down their negation.

(a) Some swans are black.
True, because I've seen a black swan
Negation: There are no black swans.

(b) Every real number is an even integer.
False, e.g., 0.5 is not an integer.
Negation: Some real number is not an even integer.

(c) 2 is even, and 3 is even.
False, because 3 is not even.
Negation using de Morgan's Law: 2 is odd or 3 is odd.

(d) If x is an even integer, then $x+1$ is odd.
True, because if 2 divides x, then 2 does not divide $x+1$.
Negation: x is even and $x+1$ is not odd.

(e) $2x = 1$
Not a statement since neither true nor false.

(3) [1, Section 2.3]: Exercises 2,3,4,5,10

Solution.

2. If a function is differentiable, then it's continuous.
3. If a function is continuous, then it's integrable.
4. If a function is polynomial, then it's rational.
5. If an integer is divisible by 8, then it's divisible by 4.
10. If the discriminant is negative, then the quadratic equation has no real solution.

(4) Are the given statements true? Formulate their negations.

(a) Not all sides of a triangle have the same length or all its angles are equal.
True, because if all sides are equal, then all angles are equal as well.
Negation: All sides of a triangle have the same length and not all angles are equal.

(b) If the integer x is a multiple of 6, then x is even.
True, because if 6 divides x, then so does 2.
Negation (Assumption of the if-then statement holds but not the conclusion): x is a multiple of 6 and x is not even.

(c) $x \in \mathbb{R}$ is a square $\Rightarrow x \geq 0$
True.
Negation: $x \in \mathbb{R}$ is a square $\wedge$ $x < 0$

(d) $2^n + 1$ is a prime number for every $n \in \mathbb{N}$.
False, because $2^3 + 1 = 9$ is not prime.
Negation: $2^n + 1$ is not prime for some $n \in \mathbb{N}$.

(e) There exists an even prime.
True, because e.g. 2 is an even prime.
Negation: There does not exist an even prime. All primes are odd.

(5) Use truth tables to show that the following hold for all logical statements P, Q, R:

(a) $P \vee (P \wedge Q) = P$

(b) $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$

Solution

(a)

P	Q	$P \vee (P \wedge Q)$
T	T	T
T	F	T
F	T	F
F	F	F

Since column 1 and 3 correspond, statements are equal.

(b)

P	Q	R	$P \wedge (Q \vee R)$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Since the last 2 columns correspond, statements are equal.

(6) Are the following equalities true for all statements P, Q? Consider truth tables.
  (a) $P \Rightarrow Q \ = \sim P \vee Q$
  (b) $\sim (P \Leftrightarrow Q) \ = \ \sim P \Leftrightarrow \ Q \ = \ P \Leftrightarrow \sim Q$

Solution

(a) True since

P	Q	$P \Rightarrow Q$	$\sim P \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

(b) True since the last 3 columns in the next table are all equal

P	Q	$P \Leftrightarrow Q$	$\sim (P \Leftrightarrow Q)$	$\sim P \Leftrightarrow Q$	$P \Leftrightarrow \sim Q$
T	T	T	F	F	
T	F	F	T	T	
F	T	F	T	T	
F	F	T	F	F	

References

[1] Richard Hammack. The Book of Proof. Creative Commons, 3rd edition, 2018. Available for free: `http://www.people.vcu.edu/~rhammack/BookOfProof/`