Math 2001 - Assignment 2

Due September 11, 2020

(1) Describe the following using set builder notation (either Axiom of Specification or Replacement):
 (a) $A =$ the set of points in $\mathbb{R}^2$ on the line through $(2,3)$ that is parallel to the y-axis
 (b) $B =$ the set of points $(x,y) \in \mathbb{R}^2$ on the line through $(1,2)$ and $(3,4)$
 (c) $C =$ the set of points in $\mathbb{R}^2$ that lie on a circle with center $(0,0)$ and radius 2

Solution.
$A = \{(x,y) \in \mathbb{R}^2 \ : \ x = 2\} = \{(2,y) \ : \ y \in \mathbb{R}\}$
$B = \{(x,y) \in \mathbb{R}^2 \ : \ -x + y = 1\}$ by finding the equation of the line through $(1,2)$ and $(3,4)$
$C = \{(x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 4\}$

(2) In the universe $U := \mathbb{N}$, compute for subsets $A = \{1,2,3,4,5\}$, $B = \{x \in U : x$ is even $\}$, and $C = \{x \in U : x \geq 4\}$:
(a) $A \cap \bar{B}$ (b) $A \cup (B \cap C)$ (c) $(A - B) \cup B$

Solution.
$A \cap \bar{B} = \{1,3,5\}$
$A \cup (B \cap C) = \{1,2,3,4,5,6,8,10,12,\dots\}$
$(A - B) \cup B = A \cup B = \{1,2,3,4,5,6,8,10,12,\dots\}$

(3) Are the following true for all sets A, B in a universe U?
 (a) $A - B = B - A$
 (b) $A \cup B \subseteq (A \cap \bar{B}) \cup (B \cap \bar{A})$
Consider Venn diagrams first and then either write a proof that the equations hold or give an example where they fail.

Solution.
 (a) $A - B = B - A$ does not hold for all sets A, B. One counterexample is $A = \{1\}, B = \emptyset$
 (b) $A \cup B \subseteq (A \cap \bar{B}) \cup (B \cap \bar{A})$ does not hold for all A, B. One counterexample is $A = B = \{1\}$.

(4) Show that for all sets A, B, C

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

without Venn diagrams.
 Recall that we already showed that the lefthand side is contained in the the right-hand side. So it only remains to write a proof for the converse,

$$(A \cup B) \cap C \supseteq (A \cap C) \cup (B \cap C).$$

Solution.
Let $x \in (A \cap C) \cup (B \cap C)$. By the definition of $\cup$ we have $x \in (A \cap C)$ or $x \in (B \cap C)$ and hence 2 cases to consider:

Case 1, $x \in (A \cap C)$**:** Then $x \in A$ and $x \in C$ by the definition of $\cap$. Since $x \in A$, we also have $x \in A \cup B$. Together with $x \in C$ this implies that $x \in (A \cup B) \cap C$.

Case 2, $x \in (B \cap C)$**:** Similar to case 1.

In either case $x \in (A \cup B) \cap C$. Hence we proved that $(A \cap C) \cup (B \cap C) \subseteq (A \cup B) \cap C$ $\square$

(5) Show for all sets A, B in the universe U:

$$\overline{A \cup B} = \bar{A} \cap \bar{B} \qquad \text{(de Morgan's law)}$$

First use Venn diagrams. Then write down a proof.

Solution.

Proof of $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$: Let $x \in \overline{A \cup B}$. Then $x \notin A \cup B$, which yields that $x \notin A$ and $x \notin B$. Hence $x \in \bar{A}$ and $x \in \bar{B}$. Thus $x \in \bar{A} \cap \bar{B}$. $\square$

Proof of $\overline{A \cup B} \supseteq \bar{A} \cap \bar{B}$: Let $x \in \bar{A} \cap \bar{B}$. Then $x \in \bar{A}$ and $x \in \bar{B}$. Equivalently $x \notin A$ and $x \notin B$. But then $x \notin A \cup B$, which yields $x \in \overline{A \cup B}$. $\square$

(6) Simplify the following sets and justify your answers:

(a) $\bigcup_{n \in \mathbb{N}} (0, n]$ (b) $\bigcap_{n=1}^{3} \{nz : z \in \mathbb{Z}\}$ (c) $\bigcup_{A \in P(\mathbb{N})} A$

In (a) we have $(0, n] = \{x \in \mathbb{R} \ : \ 0 < x \leq n\}$, the real interval from 0 to n that does not contain 0 but contains n.

Solution.

(a) $\bigcup_{n \in \mathbb{N}} (0, n] = (0, 1] \cup (0, 2] \cup (0, 3] \cup \cdots = \{x \in \mathbb{R} \ : \ x > 0\}$

These sets are equal because every $x \in (0, n]$ for some $n \in \mathbb{N}$ is also in the set on the right hand side. Conversely let $x \in \mathbb{R}$ such that $x > 0$. Then there exists $n \in \mathbb{N}$ such that $x \in (0, n]$. Hence x is in the set on the left hand side.

(b) $\bigcap_{n=1}^{3} \{nz : z \in \mathbb{Z}\}$
$= \mathbb{Z} \cap \{\ldots, -2, 0, 2, 4, 6 \ldots\} \cap \{\ldots, -3, 0, 3, 6, 9 \ldots\}$
$= \{\ldots, -6, 0, 6, 12, 18, \ldots\}$
$= \{6z : z \in \mathbb{Z}\}$

(c) $\bigcup_{A \in P(\mathbb{N})} A = \emptyset \cup \{1\} \cup \{2\} \cup \cdots \cup \{1, 2\} \cup \cdots \cup \mathbb{N} = \mathbb{N}$

The union of all subsets of $\mathbb{N}$ just yields $\mathbb{N}$.