Math 2001 - Review Problems

The following solution sketches are not complete but just give an idea of the proofs and computations for checking correctness.

(1) Show for all sets A, B:

$$P(A) \cup P(B) \subseteq P(A \cup B)$$

Proof sketch. Recall that $P(A)$ is the power set of A. Let $X \in P(A) \cup P(B)$. Then X is a subset of A or of B. Hence $X \subseteq A \cup B$. □

(2) (a) Negate the following statements:
  (i) $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathbb{R} \colon |x| < \delta \Rightarrow x^2 < \varepsilon$
  (ii) $x^y \in \mathbb{Q}$ iff $x \in \mathbb{Q}$ and $y \in \mathbb{N}$.

Solution.
  (i) Just switch $\forall$ to $\exists$ and conversely. Then negate $P \Rightarrow Q$ as $P \wedge \sim Q$.
  (ii) Negate $P \Leftrightarrow Q$ as $\sim P \Leftrightarrow Q$.

(b) Are $P \Rightarrow \sim Q$ and $\sim (P \wedge Q)$ logically equivalent?

Solution. Yes by truthtable.

(3) A word is a list of letters over the alphabet $A, \ldots, Z$.
How many 5-letter words are there that start with A or end with Z?

Solution.
Number of words starting with A: $1 * 26^4$
Number of words ending with Z: $1 * 26^4$
Number of words starting with A and ending with Z: $1^2 * 26^3$
By inclusion-exclusion there are $2 * 1 * 26^4 - 1^2 * 26^3$ words starting with A or ending with Z.

(4) Prove by contradiction: $x^2 - y^2 = 1$ has no solutions $x, y \in \mathbb{N}$. (Recall $0 \notin \mathbb{N}$ and consider a factorization of $x^2 - y^2$.)

Proof. Seeking a contradiction, suppose $x^2 - y^2 = 1$ for some $x, y \in \mathbb{N}$. Since $x^2 - y^2 = (x - y)(x + y)$, this yields that $(x - y)(x + y) = 1$. Hence $x - y = 1$ and $x + y = 1$. But this only works for $x = 1$ and $y = 0$. So $y \notin \mathbb{N}$. Contradiction.

Thus our assumption was wrong. There are no such $x, y \in \mathbb{N}$. □

(5) Show for any $n \in \mathbb{N}$ with $n \geq 3$:

$$\sum_{i=2}^{n-1} \binom{i}{2} = \binom{n}{3}.$$

Proof by induction.
Base case: Check for $n = 3$.
Induction assumption: $\sum_{i=2}^{n-1} \binom{i}{2} = \binom{n}{3}$ for a fixed $n \in \mathbb{N}$.
Induction step: Replace n by $n+1$ in the formula and show $\sum_{i=2}^{n} \binom{i}{2} = \binom{n+1}{3}$.
Start on the lefthand side and use the induction assumption

$$\sum_{i=2}^{n} \binom{i}{2} = \sum_{i=2}^{n-1} \binom{i}{2} + \binom{n}{2} = \binom{n}{3} + \binom{n}{2} = \cdots = \binom{n+1}{3}$$

□

(6) (a) The kernel of a function $f : A \to B$ is the relation

$$R = \{(x, y) \in A^2 \ : \ f(x) = f(y)\}.$$

Show that R is an equivalence relation on A.

(b) For $x, y \in \mathbb{Z}_6$ let

$$x \, S \, y \text{ if } x^2 = y^2.$$

Then S is an equivalence relation on $\mathbb{Z}_6$. Describe the equivalence classes of S. How many are there?

Solution.
(a) Check that R is reflexive, symmetric, transitive.
(b) Denote the elements of $\mathbb{Z}_6$ by $0, \ldots, 5$ (no brackets).
Two elements are equivalent if their squares are equal. So consider the squares $0^2 = 0, 1^2 = 1, 2^2 = 4, 3^2 = 3, 4^2 = 4, 5^2 = 1$
Hence the class of 0 is $\{0\}$ since 0 is only related to itself, the class of 1 is $\{1, 5\}$ since 1 and 5 are related,
⋮
4 different classes in total.

(7) Solve $[9] \cdot [x] = [5]$ in $\mathbb{Z}_{43}$ using the extended Euclidean algorithm.

Solution. Solve $9x + 43y = 5$ for integers x, y.

First compute $\gcd(43, 9)$ with Bezout coefficients u, v such that $43u + 9v = 1$. Then multiply this Bezout identity with 5 to get $43(5u) + 9(5v) = 5$. Hence $x = 5v$ is a solution.

(8) (a) Check whether the function

$$f\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}, (x,y)\mapsto x+2y,$$

is injective, surjective, bijective.

Solution. Not injective, but surjective. Hence not bijective.

(b) Let $f\colon A\to B$ and $g\colon B\to C$. Prove that if $g\circ f$ is injective, then f is injective.

Solution. Use contraposition (Assignment on HW 13).