Review 2: Logic

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Implications

Let P, Q be statements.

statement	equivalent meanings	negation
$P \Rightarrow Q$	If P is true, then Q is true. $\sim P \vee Q$ $\sim Q \Rightarrow \sim P$ (contrapositive)	$P \wedge \sim Q$
$P \Leftrightarrow Q$	Q is true if and only if P is true. $(P \Rightarrow Q) \wedge (P \Leftarrow Q)$	$P \Leftrightarrow \sim Q$ $\sim P \Leftrightarrow Q$

How to prove $P \Rightarrow Q$:

- **direct:** Assume P. Show Q.
- **contrapositive:** Assume $\sim Q$. Show $\sim P$.
- **by contradiction** (only as last resort): Assume P and $\sim Q$. Show some contradiction.

How to prove $P \Leftrightarrow Q$:

Show $P \Rightarrow Q$ and show $P \Leftarrow Q$.

Equivalent statements

Theorem

Let $f\colon A \to B$ be a function. Then the following are equivalent (TFAE):

1. f is bijective.
2. $\forall y \in B\ \exists$ unique $x \in A:\ f(x) = y$.
3. f has an inverse function.
4. $\exists \ell, r\colon B \to A:\ \ell \circ f = \mathrm{id}_A$ and $f \circ r = \mathrm{id}_B$.

This Theorem states that 1. $\Leftrightarrow$ 2. $\Leftrightarrow$ 3. $\Leftrightarrow$ 4.
Instead of showing 3 times $\Leftrightarrow$, such statements can be proved more efficiently as a cycle of implications:

$$1. \Rightarrow 2. \Rightarrow 3. \Rightarrow 4. \Rightarrow 1.$$

Quantified statements

Let A be a set, $P(x)$ be a statement for $x \in A$.

statement	meaning	negation
$\forall x \in A\colon P(x)$	For all $x \in A$, $P(x)$ is true.	$P(x)$ is not true for some $x \in A$. $\exists x \in A\colon \sim P(x)$
$\exists x \in A\colon P(x)$	There exists $x \in A$ such that $P(x)$ is true.	$P(x)$ is not true for all $x \in A$. $\forall x \in A\colon \sim P(x)$

How to prove $\forall x \in A\colon P(x)$

Let $x \in A$ (arbitrary but fixed). Show $P(x)$.

How to refute $\forall x \in A\colon P(x)$

Give concrete explicit $x \in A$ that does not satisfy $P(x)$.

How to prove $\exists x \in A\colon P(x)$

Give a concrete explicit $x \in A$ that satisfies $P(x)$.

How to refute $\exists x \in A\colon P(x)$

Show $\forall x \in A\colon \sim P(x)$.

Quantifiers and implications in Calculus

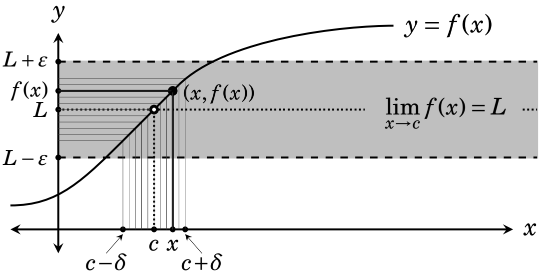
Calculus: Informally $\lim_{x\to c} f(x) = L$ means that $f(x)$ is arbitrarily close to L provided that x is sufficiently close to c. More precisely

Definition (Limit of a function)

Let $c \in \mathbb{R}$ and $f : \mathbb{R} - \{c\} \to \mathbb{R}$. Then $\lim_{x\to c} f(x) = L$ if

$$\forall \varepsilon \in \mathbb{R}^+ \; \exists \delta \in \mathbb{R}^+ \; \forall x \in \mathbb{R} - \{c\} : \; |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Diagram taken from Hammack, Book of Proof, 2018.



How to prove $\forall \varepsilon \in \mathbb{R}^+ \ \exists \delta \in \mathbb{R}^+ \ \forall x \in \mathbb{R} - \{c\}$: $|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Example

Prove $\lim_{x \to 0} 2x \sin(\frac{1}{x}) = 0$.

- ($\forall$) Let $\varepsilon > 0$ arbitrary, fixed for the remainder of the proof.
- ($\exists$) Find $\delta > 0$ (δ may depend on ε) such that $\forall x \neq c$:
$$|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon. \qquad (*)$$
- ($\forall$) Let $x \in \mathbb{R} - \{0\}$ arbitrary, fixed. For (*) consider

$$|f(x) - L| = \left| 2x \sin(\frac{1}{x}) - 0 \right| = 2|x| \underbrace{|\sin(\frac{1}{x})|}_{\leq 1} \leq 2|x|$$

- The latter is $< \varepsilon$ if $|x| < \varepsilon/2$.
- We found $\delta := \varepsilon/2$ such that $|x - 0| < \delta \Rightarrow |f(x) - 0| < \varepsilon$.
- Thus $\lim_{x \to 0} 2x \sin(\frac{1}{x}) = 0$. □

How to prove a limit does not exist?

We need to show that for all $L \in \mathbb{R}$,

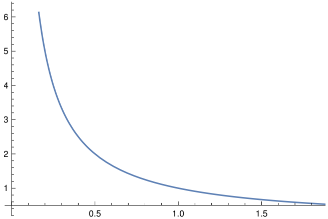
$$\sim (\forall \varepsilon \in \mathbb{R}^+ \; \exists \delta \in \mathbb{R}^+ \; \forall x \in \mathbb{R} - \{c\} : \; |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon),$$

equivalently,

$$\forall L \in \mathbb{R} \; \exists \varepsilon \in \mathbb{R}^+ \; \forall \delta \in \mathbb{R}^+ \; \exists x \in \mathbb{R} - \{c\} : \; |x - c| < \delta \text{ and } |f(x) - L| \geq \varepsilon.$$

Example

Prove $\lim_{x \to 0} \frac{1}{x}$ does not exist.



- ($\forall$) Let $L \in \mathbb{R}^+$, ($\exists$) guess $\varepsilon > 0$, ($\forall$) let $\delta > 0$ arbitrary.
- ($\exists$) Find $x \in \mathbb{R} - \{0\}$: $|x - 0| < \delta$ and $|\frac{1}{x} - L| \geq \varepsilon$.
- Say $0 < x < \delta$ and $\frac{1}{x} - L \geq \varepsilon$. Hence $\frac{1}{\varepsilon + L} \geq x$ and e.g. $x := \min(\frac{\delta}{2}, \frac{1}{\varepsilon + L})$ works!
- Thus $\lim_{x \to 0} \frac{1}{x}$ cannot be any $L \in \mathbb{R}^+$. $\square$

Sum rule for limits

$\lim_{x \to c} f(x) = L$ if $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \neq c\colon\ |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Theorem

If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ both exist, then

$$\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x).$$

Proof.

- Let $\lim_{x \to c} f(x) =: L$ and $\lim_{x \to c} g(x) =: M$. Show $\lim_{x \to c} [f(x) + g(x)] = L + M$.
- Let $\varepsilon > 0$ arbitrary. Find $\delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) + g(x) - L - M| < \varepsilon$.
- $|f(x) + g(x) - L - M| \leq |f(x) - L| + |g(x) - M|$ by the triangle inequality.
- We have $\delta_1 > 0\colon |x - c| < \delta_1 \Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$,
  $\delta_2 > 0\colon |x - c| < \delta_2 \Rightarrow |g(x) - M| < \frac{\varepsilon}{2}$.
- Then $|x - c| < \underbrace{\min(\delta_1, \delta_2)}_{=:\delta} \Rightarrow |f(x) + g(x) - L - M| < \underbrace{\frac{\varepsilon}{2} + \frac{\varepsilon}{2}}_{=\varepsilon}$.

- Do you want to know more? Take Math 3001 – Analysis