Functions 4

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Inverse functions

Inverses

Example

For $f\colon \mathbb{R} \to \mathbb{R}^+,\ x \mapsto e^x$ and $g\colon \mathbb{R}^+ \to \mathbb{R},\ x \mapsto \log x$,

$g \circ f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto \log e^x \ = x$
$f \circ g\colon \mathbb{R}^+ \to \mathbb{R}^+,\ x \mapsto e^{\log x} \ = x$

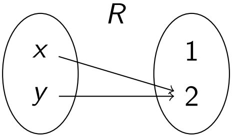
Here f and g undo each other (are inverses of each other).

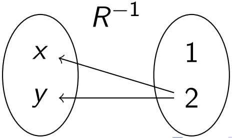
Definition

The **inverse relation** of $R \subseteq A \times B$ is

$$R^{-1} := \{(b, a) \ : \ (a, b) \in R\}.$$

R^{-1} is a relation from B to A but usually not a function (even if R is a function).





Inverses and bijectivity

Theorem

$f\colon A \to B$ is bijective iff its inverse relation f^{-1} is a function.

Proof.

$\Rightarrow$ Assume f is bijective.

- Since f is surjective, $\forall b \in B\ \exists a \in A\colon f(a) = b$.
- This a is unique since f is injective.
- Hence $\forall b \in B\ \exists$ unique $a \in A\colon \underbrace{(a, b) \in f}_{(b,a)\in f^{-1}}$.

Thus f^{-1} is a function from B to A.

$\Leftarrow$ Assume f^{-1} is a function.

- Then $\forall b \in B\ \exists$ unique $a \in A\colon \underbrace{(b, a) \in f^{-1}}_{(a,b)\in f}$.

Thus f is surjective and injective. □

Definition

Let $f\colon A \to B$ be bijective. Then $f^{-1}\colon B \to A$ is the **inverse function** of f.

Let $\mathrm{id}_A\colon A \to A, x \mapsto x$, denote the **identity map** on A.

Lemma

Let $f\colon A \to B$ be bijective. Then $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$.

Proof.

HW $\square$

Note (to be proved in 2 slides).

If there exist functions $l, r\colon B \to A$ that act like the inverse when composed with $f\colon A \to B$ on either side, then they **are** f^{-1}. Hence instead of checking that f is bijective, it suffices to find f^{-1} (and check that $f^{-1} \circ f = \mathrm{id}_A$, $f \circ f^{-1} = \mathrm{id}_B$).

Example

Find the inverse of $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^3 + 1$, if it exists.

1. $f^{-1} = \{(x^3 + 1, x) \ : \ x \in \mathbb{R}\}$ is not very useful.
2. Instead solve $y = f(x)$ for x.
   - $y = x^3 + 1$ yields $\sqrt[3]{y - 1} = x$.
   - $f^{-1} : \mathbb{R} \to \mathbb{R}, \ y \mapsto \sqrt[3]{y - 1}$

Note that $f^{-1} \circ f = \mathrm{id}_{\mathbb{R}}, \ f \circ f^{-1} = \mathrm{id}_{\mathbb{R}}$.

If it behaves like the inverse, it is the inverse

Lemma

Let $f\colon A \to B$, and $l, r\colon B \to A$ such that $l \circ f = \mathrm{id}_A$ and $f \circ r = \mathrm{id}_B$. Then f is bijective and $l = r = f^{-1}$.

Proof.

- Since $l \circ f = \mathrm{id}_A$ is injective, f is injective by a previous Thm.
- Since $f \circ r = \mathrm{id}_B$ is surjective, f is surjective by previous Thm.
- So f is bijective and has an inverse f^{-1}.
- To show that $l = f^{-1}$ consider
  $l \circ (f \circ f^{-1}) = l \circ \mathrm{id}_B = l$
  $(l \circ f) \circ f^{-1} = \mathrm{id}_A \circ f^{-1} = f^{-1}$
  Since the left hand sides are the same by associativity,
  $l = f^{-1}$.
- $r = f^{-1}$ follows similarly.

□

Example

Find the inverse of $f\colon \mathbb{R} \to \mathbb{R}_0^+, x \mapsto x^2$, if it exists.

- Let $y \in \mathbb{R}_0^+$. Solve $y = f(x)$ for x.
- $\sqrt{y} = x$ yields $g\colon \mathbb{R}_0^+ \to \mathbb{R}, y \to \sqrt{y}$.
- Check that

$$\begin{aligned} g(f(x)) &= x \quad \text{for all } x \in \mathbb{R} \\ f(g(y)) &= y \quad \text{for all } y \in \mathbb{R}_0^+ \end{aligned}$$

- Hence $g = f^{-1}$ and f is bijective.

Example

Find the inverse of $f\colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto ((x^2 + 1)y, x^3)$.

- For $(u, v) \in \mathbb{R}^2$ solve $f(x, y) = (u, v)$:

$$\begin{aligned} (x^2 + 1)y &= u \\ x^3 &= v \end{aligned}$$

- Then $x = \sqrt[3]{v}$ and $y = \frac{u}{\sqrt[3]{v}^2 + 1}$.
- $f^{-1}\colon \mathbb{R}^2 \to \mathbb{R}^2, (u, v) \mapsto \left(\sqrt[3]{v}, \frac{u}{\sqrt[3]{v}^2 + 1}\right)$.

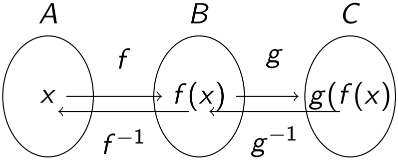
The inverse of a composition

Lemma

For $f : A \to B, g : B \to C$ bijective

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

To "undo" $g \circ f$, the inverses need to be composed in the opposite order.



Proof.

Show that $f^{-1} \circ g^{-1}$ behaves like the inverse of $f \circ g$:

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ \mathrm{id}_B \circ f = \mathrm{id}_A$$

Similarly check $(g \circ f) \circ (f^{-1} \circ g^{-1})$. □