Functions

Peter Mayr

CU, Discrete Math, November 20, 2020

Counting functions

The number of functions $f : \{1, 2, \ldots, k\} \to \{1, \ldots, n\}$

Let $k, n \in \mathbb{N}_0$.

- all functions: n^k
  [For each $i \in \{1, \ldots, k\}$ choose an image $f(i) \in \{1, \ldots, n\}$ without restrictions.]
- injective functions ($n \geq k$): $n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$
  [Choose $f(i)$ without repetitions!]
- surjective functions: ???

Example

Number of surjective functions $\{1, \ldots, k\} \to \{1, \ldots, n\}$:

1. $n = 1$, all functions are surjective: 1
2. $n = 2$, all functions minus the non-surjective ones, i.e., those that map into proper subsets $\{1\}, \{2\}$: $2^k - 1^k - 1^k$
3. $n = 3$, subtract all functions into 2-element subsets (double counting those into 1-element subsets!): $3^k - 3 \cdot 2^k + 3 \cdot 1^k$

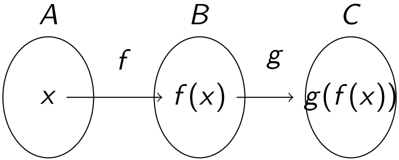
Composition of functions

Composing functions

Definition

Let $f\colon A \to B, g\colon B \to C$. The **composition** of f with g is

$$g \circ f\colon A \to C, x \mapsto g(f(x)).$$



Example

For $f\colon \mathbb{Z} \to \mathbb{Z},\ x \mapsto x+1$ and $g\colon \mathbb{Z} \to \mathbb{Z},\ x \mapsto x^2$,

$g \circ f\colon \mathbb{Z} \to \mathbb{Z},\ x \mapsto (x+1)^2$
$f \circ g\colon \mathbb{Z} \to \mathbb{Z},\ x \mapsto x^2+1$

Note the order: composition is not commutative.

Properties of function composition

Theorem (Function composition is associative)

For $f\colon A \to B, g\colon B \to C, h\colon C \to D$,

$$(h \circ g) \circ f = h \circ (g \circ f).$$

How to show equality between functions?

Proof.

Show that $\forall x \in A\colon ((h \circ g) \circ f)(x) = (h \circ (g \circ f))(x)$

Let $x \in A$. Then

$((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x))),$

$(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x))).$ $\square$

Theorem

Let $f\colon A \to B, g\colon B \to C$.

1. If $g \circ f$ is injective, then f is injective.
2. If $g \circ f$ is surjective, then g is surjective.

Proof.

By contraposition, HW $\square$