Functions 2

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The pigeonhole principle

If k things are put into n boxes and $k > n$, then at least 1 box contains 2 things (or more).

This can be formalized to functions on finite sets to see that injectivity and surjectivity pose strong conditions on the sizes of domain and codomain.

Lemma

Let A, B finite and let $f\colon A \to B$.

1. If $|A| < |B|$, then f is not surjective.
2. If $|A| > |B|$, then f is not injective (cf. pigeonhole principle).

Proof.

1. Note $|f(A)| \leq |A| < |B|$. Hence $f(A) \neq B$.
2. Contrapositive: Suppose f is injective. Then $|A| = |f(A)|$ and the latter is $\leq |B|$. □

Theorem

Let A, B finite with $|A| = |B|$ and let $f : A \to B$. TFAE:

1. f is injective.
2. f is surjective.
3. f is bijective.

Proof.

$1 \Rightarrow 2$: HW

$2 \Rightarrow 1$: HW

So $1 \Leftrightarrow 2$ which also implies $1 \Leftrightarrow 3$. □