Functions 1

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Functions are special relations

Example

$f(x) = x^2$ on $\mathbb{R}$ is determined by its graph $f = \{(x, x^2) \; : \; x \in \mathbb{R}\}$.

Definition

Let A, B be sets. A **function** f from A to B is a relation from A to B such that

$$\forall x \in A \; \exists \text{ unique } y \in B \colon (x, y) \in f.$$

Then we usually write $f(x) = y$ [*instead of* $(x, y) \in f$] and

$$f \colon A \to B, \;\; x \mapsto f(x).$$

[*Read: f is the function from A to B that maps x to $f(x)$.*]

Example

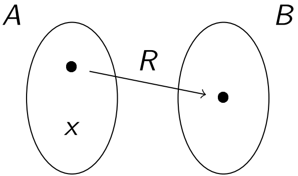
$R = \{(y^2, y) \; : \; y \in \mathbb{R}\}$ is not a function

- from $\mathbb{R}$ to $\mathbb{R}$ because $\nexists y \in \mathbb{R} \colon (-2, y) \in R$;
- from $\mathbb{R}_0^+$ to $\mathbb{R}$ because $(1, -1), (1, 1) \in R$.

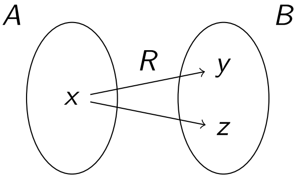
Failing to be a function

A relation $R \subseteq A \times B$ is **not a function** if

1. some $x \in A$ is not paired up with any $y \in B$;



2. some $x \in A$ is paired up with more than one element in B.



Definition

A function $f\colon A \to B$ has

- **domain** A,
- **codomain** B,
- **image** (or **range**)

$$f(A) := \{f(x) \;:\; x \in A\}.$$

Note: the image is a subset of the codomain, $f(A) \subseteq B$.

Example

1. $f\colon \mathbb{R} \to \mathbb{R},\; x \mapsto x^2$, has domain and codomain $\mathbb{R}$, image $\mathbb{R}_0^+$. [Every nonnegative number y can be written as a square, $y = f(\sqrt{y}) \in f(\mathbb{R})$.]
2. $g\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N},\; (x, y) \mapsto 2^x 3^y$, has domain $\mathbb{N} \times \mathbb{N}$, codomain $\mathbb{N}$, image $g(\mathbb{N} \times \mathbb{N}) = \{2^x 3^y \;:\; x, y \in \mathbb{N}\}$.

Images and preimages

Let $f\colon A \to B$.

1. For $X \subseteq A$, the **image of** X is
$$f(X) := \{f(x) \ : \ x \in X\}.$$
2. For $Y \subseteq B$, the **preimage of** Y is the set of all elements that f maps into Y,
$$f^{-1}(Y) := \{x \in A \ : \ f(x) \in Y\}.$$

Example

$f\colon \mathbb{R} \to \mathbb{R}, \ x \mapsto x^2$

- $f(\{0,1,2\}) = \{0,1,4\}$ $\quad$ $f([2,5]) = [4,25]$
- $f^{-1}(\{0,1,4\}) = \{-2,-1,0,1,2\}$ $\quad$ $f([9,36]) = [-6,3] \cup [3,6]$

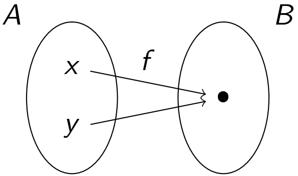
Injective functions

Definition

$f\colon A \to B$ is **injective** (one-to-one) if

$$\forall x, y \in A\colon f(x) = f(y) \Rightarrow x = y.$$

- Equivalently, $\forall x, y \in A\colon x \neq y \Rightarrow f(x) \neq f(y)$
(distinct elements are mapped to distinct elements)
- Pictorially f is **not injective** if



Example

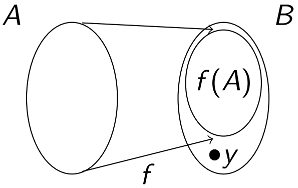
$f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto x^2$, is not injective since $f(2) = f(-2)$.

Surjective functions

Definition

$f: A \to B$ is **surjective** (onto B) if $\forall y \in B \; \exists x \in A$: $f(x) = y$.

- Equivalently $f(A) = B$ image of f is equal to its codomain
- Pictorially f is **not surjective** if



Example

- $f: \mathbb{R} \to \mathbb{R}, \; x \mapsto x^2$, is not surjective since $-2 \in f(\mathbb{R})$.
- $g: \mathbb{R} \to \mathbb{R}_0^+, \; x \mapsto x^2$, is surjective since $g(\mathbb{R}) = \mathbb{R}_0^+$.

$f: A \to B$ becomes surjective by restricting its codomain to $f(A)$.

Injective and surjective = bijective

$f\colon A \to B$ is

1. **injective** if $\forall x, y \in A\colon f(x) = f(y) \Rightarrow x = y$,
2. **surjective** if $\forall y \in B\ \exists x \in A\colon f(x) = y$,
3. **bijective** if injective and surjective.

Example

$f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}, x \mapsto \frac{1}{x}$

1. injective: Let $x, y \in \mathbb{R} \setminus \{0\}$ such that $\frac{1}{x} = \frac{1}{y}$. Multiplying by xy yields $y = x$.
2. not surjective: $0 \notin f(\mathbb{R} \setminus \{0\})$ since $\frac{1}{x} \neq 0$ for any $x \in \mathbb{R} \setminus \{0\}$. In fact $f(\mathbb{R} \setminus \{0\}) = \mathbb{R} \setminus \{0\}$.
3. not bijective since not surjective

The restriction $g\colon \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\},\ x \mapsto \frac{1}{x}$, is bijective

Example

$f : \mathbb{R}^2 \to \mathbb{R}^2, \ (x, y) \mapsto (x + y, x - y)$

1. injective: Let $(x, y), (u, v) \in \mathbb{R}^2$ s.t. $f(x, y) = f(u, v)$, i.e.
$$x + y = u + v$$
$$x - y = u - v$$
Adding/subtracting these equations yields
$$2x = 2y$$
$$2y = 2v$$
Hence $(x, y) = (u, v)$ and f is injective.
2. surjective: Let $(u, v) \in \mathbb{R}^2$. Find (x, y) s.t. $f(x, y) = (u, v)$.
Solve
$$\begin{array}{rl} x + y & = u \\ x - y & = v \\ \hline 2x \qquad & = u + v \\ 2y & = u - v \end{array}$$
Then $x = \frac{u+v}{2}, y = \frac{u-v}{2}$ yield $f(x, y) = (u, v)$. So f is surjective.
3. f is bijective since injective and surjective.