FUNCTIONS

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1. Basic properties

Definition. Let A, B be sets. A **function** f from A to B is a relation such that

$$\forall x \in A\ \exists! y \in B\colon (x, y) \in f,$$

(for every $x \in A$ there exists exactly one $y \in B$ such that $(x, y) \in f$). Then we usually write $f(x) = y$ and $f\colon A \to B,\ x \mapsto f(x)$.

A is the **domain**, B is the **codomain**, and

$$f(A) := \{f(x)\ :\ x \in A\}$$

is the **range** (or **image**) of f.

Definition. A function $f\colon A \to B$ is

(1) **injective** (one-to-one) if $\forall x, y \in A\colon f(x) = f(y) \Rightarrow x = y$
(2) **surjective** (onto) if $\forall y \in B\ \exists x \in A\colon f(x) = y$
(3) **bijective** if injective and surjective.

Remark. Equivalently $f\colon A \to B$ is

(1) **injective** if $\forall x, y \in A\colon x \neq y \Rightarrow f(x) \neq f(y)$ (contraposition),
(2) **surjective** if $f(A) = B$.

2. Pigeonhole principle

For functions on finite sets, injectivity and surjectivity pose strong conditions on the sizes of domain and codomain.

Lemma 1. *Let A, B finite with $|A| = |B$ and let $f\colon A \to B$. TFAE:*

(1) *f is injective.*
(2) *f is surjective.*
(3) *f is bijective.*

Lemma 2. *Let A, B finite and let $f\colon A \to B$.*

(1) *If $|A| > |B|$, then f is not injective.*
(2) *If $|A| < |B|$, then f is not surjective.*

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3. Composition

Definition. Let $f\colon A \to B, g\colon B \to C$. The **composition** of f with g is

$$g \circ f\colon A \to C, x \mapsto g(f(x)).$$

Composition of functions is not commutative but associative.

Theorem 3. *The composition of injective functions is injective. The composition of surjective functions is surjective.*

4. Inverse functions

Definition. The **inverse relation** of $R \subseteq A \times B$ is

$$R^{-1} := \{(b,a) \ : \ (a,b) \in R\}.$$

R^{-1} is a relation from B to A but usually not a function (even if R is a function).

Theorem 4. *A function $f\colon A \to B$ is bijective iff the inverse relation f^{-1} is a function from B to A.*

Definition. Let $f\colon A \to B$ be bijective. Then $f^{-1}\colon B \to A$ is the **inverse function** of f.

Let $\mathrm{id}_A\colon A \to A, x \mapsto x$, denote the **identity map** on A.

Lemma 5. *Let $f\colon A \to B$ be bijective. Then $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$.*

If there exist functions l, r that act like the inverse when composed with f on either side, then they **are** f^{-1}. Hence instead of checking that f is bijective, it suffices to find f^{-1} (and check that $f^{-1} \circ f = \mathrm{id}_A$, $f \circ f^{-1} = \mathrm{id}_B$).

Lemma 6. *Let $f\colon A \to B$, and $l, r\colon B \to A$ such that $l \circ f = \mathrm{id}_A$ and $f \circ r = \mathrm{id}_B$. Then f is bijective and $l = r = f^{-1}$.*

5. Image and preimage

Definition. Let $f\colon A \to B$.

1. For $X \subseteq A$, the **image** of X is $f(X) := \{f(x) \ : \ x \in X\}$.
2. For $Y \subseteq B$, the **preimage** of Y is $f^{-1}(Y) := \{x \in A \ : \ f(x) \in Y\}$.

Note that f is not necessarily bijective and f^{-1} is not necessarily a function. $f^{-1}(Y)$ denotes the set of all elements that f maps into Y.