Equivalence relations

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Recall

Definition

A relation $R \subseteq A \times A$ is an **equivalence** on A if R is

1. reflexive, $\forall x \in A \colon xRx$
2. symmetric, $\forall x, y \in A \colon xRy \Rightarrow yRx$
3. transitive. $\forall x, y, z \in A \colon (xRy \wedge yRz) \Rightarrow xRz$

Note

1. Equivalence relations are used for classifying 'similar' elements of A.
2. They are often denoted by symbols like $\sim, \approx, \equiv, \cong \ldots$
3. Examples are $=, \equiv_n$ on $\mathbb{Z} (n \in \mathbb{N})$, having the same absolute value on $\mathbb{R}$, having the same cardinality on sets, the relation of all pairs on A, belonging to the same family, looking the same, ...

Class of a = everything related to a

Definition

For an equivalence relation $\sim$ on A and $a \in A$,

$$[a]_\sim := \{x \in A \ : \ x \sim a\} \qquad \text{(set of all elements equivalent to a)}$$

is the **equivalence class** of a with respect to $\sim$.

Example

Every $x \in \mathbb{Z}$ is either $\equiv_2 0$ or $\equiv_2 1$.

$$[0] = \{2z \ : \ z \in \mathbb{Z}\} \qquad = [2] = [-2] = [4] = \dots$$
$$[1] = \{2z+1 \ : \ z \in \mathbb{Z}\} \quad = [-1] = [3] = \dots$$

Note

$[0] \cup [1] = \mathbb{Z}$ (each element is in some class)

$[0] \cap [1] = \emptyset$ (no element is in 2 distinct classes)

each element is in exactly one class

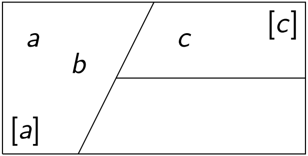
Equivalence classes

Theorem (Partition into classes)

Let $\sim$ be an equivalence on A, let $a, b \in A$. Then

1. $[a] = [b]$ iff $a \sim b$;
2. $[a] \cap [b] = \emptyset$ iff $a \not\sim b$;
3. $\bigcup_{a \in A} [a] = A$.

The set A is partitioned into disjoint equivalence classes:



Proof 1.

$\Rightarrow$: Assume $[a] = [b]$. Show $a \sim b$.

- Note $a \in [a]$ by the reflexivity of $\sim$.
- So $a \in [b] = \{x \in A \ : \ x \sim b\}$ and $a \sim b$.

Proof 1.

$\Leftarrow$: Assume $a \sim b$. Show $[a] = [b]$.

To show $[a] \subseteq [b]$, let $c \in [a]$.

- Then $c \sim a$ and $a \sim b$ yield $c \sim b$ since $\sim$ is transitive.
- So $c \in [b]$.

To show $[b] \subseteq [a]$, let $c \in [b]$.

- Then $c \sim b$ and $b \sim a$ since $\sim$ is symmetric.
- Transitivity implies $c \sim a$.
- So $c \in [a]$.

Thus $[a] = [b]$. $\square$

Proof 2.

Show $[a] \cap [b] = \emptyset$ iff $a \not\sim b$.

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Proof 3.

Show $\bigcup_{a \in A} [a] = A$.

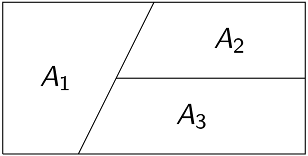
- $\bigcup_{a \in A} [a] \subseteq A$ holds since $[a] \subseteq A$ for all $a \in A$.
- To show $A \subseteq \bigcup_{a \in A} [a]$, let $x \in A$.
  Since $x \in [x]$ by reflexitivity, $x \in \bigcup_{a \in A} [a]$. $\square$

Partitions

Definition

A **partition** of a set A is a set of non-empty subsets $\{A_i \;:\; i \in I\}$ such that

1. $A_i \cap A_j = \emptyset$ for all distinct $i, j \in I$,
2. $\bigcup_{i \in I} A_i = A$.



Each element of A is in exactly one subset A_i.

Any equivalence on A gives a partition of A and conversely

Corollary

Let $\sim$ be an equivalence on A. Then the set of equivalence classes $\{[a] \ : \ a \in A\}$ is a partition of A.

Proof.

The defining conditions of a partition follow immediately from the previous Theorem. □

Theorem

Let $\{A_i \ : \ i \in I\}$ a partition of A. For $a, b \in A$ define

$$a \sim b \text{ if } a, b \in A_i \text{ for some } i \in I.$$

Then $\sim$ is an equivalence relation on A with classes $\{A_i \ : \ i \in I\}$.

Proof.

HW □