Cardinality of sets

Peter Mayr

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When do two sets have the same size?

Recall

For finite sets A, B there exists a bijection $f\colon A \to B$ iff $|A| = |B|$.

This motivates the following general definition.

Definition

Sets A and B have the same **cardinality**, written $|A| = |B|$, if there exists a bijection $f\colon A \to B$.

Hilbert's Hotel

Imagine a hotel with infinitely many rooms numbered $1, 2, 3, \ldots$
All rooms are occupied. How to find space for a new guest?
Tell each old occupant to move one room down (from 1 to 2, from 2 to 3, ...).
Then room 1 becomes free and all old guests still have a room.

$f\colon \mathbb{N} \to \mathbb{N} \setminus \{1\},\ x \mapsto x + 1$, is bijective.
Hence $|\mathbb{N}| = |\mathbb{N} \setminus \{1\}|$.

What does it mean to count? Finding a bijection with $\{1, \dots, n\}$!

Example

There is a bijection $f \colon \{a, b, c\} \to \{1, 2, 3\}$,

$$f: \quad a \mapsto 1$$
$$b \mapsto 2$$
$$c \mapsto 3$$

Definition

A set A is **finite** if there exists $n \in \mathbb{N}_0$ such that $|A| = |\{1, \dots, n\}|$; otherwise A is **infinite**.

Note

Unlike a finite set, an infinite set can have a proper subset with the same cardinality (like $\mathbb{N} \setminus \{1\} \subsetneq \mathbb{N}$).

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R} \ldots$ are all infinite

Theorem

$|\mathbb{N}| = |\mathbb{Z}|$.

Proof.

For a bijection f between $\mathbb{Z}$ and $\mathbb{N}$, send negative to even and non-negative to odd numbers:

$$f : \mathbb{Z} \to \mathbb{N}, \; x \mapsto \begin{cases} -2x & \text{if } x < 0 \\ 2x + 1 & \text{if } x \geq 0 \end{cases}$$

x	$\ldots$	-3	-2	-1	0	1	2	3	$\ldots$
$f(x)$	$\ldots$	6	4	2	1	3	5	7	$\ldots$

Claim: f is injective.

- Let $x, y \in \mathbb{Z}$ such that $f(x) = f(y)$.
- Case $f(x)$ is even: Then $f(x) = -2x = -2y$ yields $x = y$.
- Case $f(x)$ is odd: Then $f(x) = 2x + 1 = 2y + 1$ yields $x = y$.

Recall the definition

$$f : \mathbb{Z} \to \mathbb{N}, \; x \mapsto \begin{cases} -2x & \text{if } x < 0 \\ 2x + 1 & \text{if } x \geq 0 \end{cases}$$

Claim: f is surjective.

- Let $y \in \mathbb{N}$.
- Case y is even: Then $y = f(x)$ for $x = -\frac{y}{2} \in \mathbb{Z}$.
- Case y is odd: Then $y = f(x)$ for $x = \frac{y-1}{2} \in \mathbb{Z}$.

Hence f is bijective and $|\mathbb{N}| = |\mathbb{Z}|$. □

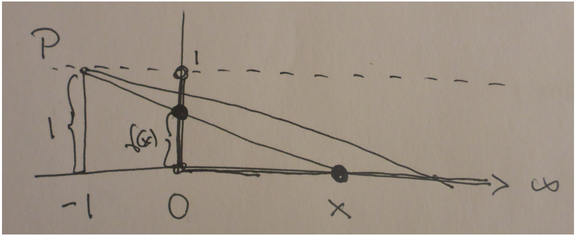
$\mathbb{R}$ is as big as the open interval $(0, 1)$

Theorem

$|\mathbb{R}| = |(0, 1)|$

Proof.

- $f : \mathbb{R}^+ \to (0, 1),\ x \mapsto \frac{x}{x+1}$, is bijective.
  This projects a point x on the positive x-axis to a point $f(x)$ between 0 and 1 on the y-axis:



- $g : \mathbb{R} \to \mathbb{R}^+,\ x \mapsto e^x$, is bijective.
- $f \circ g : \mathbb{R} \to (0, 1)$ is bijective.

Theorem

$|[0,1]| = |(0,1)|$

Proof.

HW □

There are more reals than integers

Theorem (Cantor 1891)

$|\mathbb{N}| \neq |\mathbb{R}|$

Proof (Cantor's diagonal argument).

Show that no function $f : \mathbb{N} \to \mathbb{R}$ can be surjective. Consider

n	$f(n)$
1	$*.a_1 a_2 a_3 \ldots$
2	$*.b_1 b_2 b_3 \ldots$
3	$*.c_1 c_2 c_3 \ldots$
$\vdots$	

$a_i, b_i, \ldots$ digits in decimal expansion

Let $z \in \mathbb{R}$ such that the n-th decimal place of z is distinct from the n-the decimal place of $f(n)$ for all $n \in \mathbb{N}$:

$$z = 0.z_1 z_2 z_2 \ldots \text{ with } z_1 \neq a_1,\ z_2 \neq b_2,\ z_3 \neq b_3, \ldots$$

Then $z \neq f(n)$ for all $n \in \mathbb{N}$. Hence f is not surjective. □

There are different sizes of infinite sets!

Definition

A set A is **countably infinite** if $|A| = |\mathbb{N}|$. The cardinality of $\mathbb{N}$ is $\aleph_0 := |\mathbb{N}|$ ('aleph zero', from Hebrew alphabet).
A is **uncountable** if A is infinite and $|A| \neq |\mathbb{N}|$.

Note

Every infinite set A has a countably infinite subset,

$$\begin{array}{cccccc} 1, & 2, & \dots & n, & n+1, & \dots \in \mathbb{N} \\ \downarrow & \downarrow & & \downarrow & \downarrow & \\ a_1, & a_2, & \dots & a_n, & a_{n+1}, & \dots \in A \end{array}$$

$\aleph_0$ is the smallest size an infinite set can have (the first infinite cardinal).

Example

$\mathbb{N}, \mathbb{Z}, \mathbb{N} \times \mathbb{N}, \mathbb{Q} \dots$ are countably infinite.
$\mathbb{R}, [0,1], \mathbb{C}, P(\mathbb{N}) \dots$ are uncountable.

Why countable?

Note

A is countably infinite iff its elements can be enumerated as $a_1, a_2, a_3, \ldots$

Such an enumeration is just a bijection $\mathbb{N} \to A$,
$$\begin{aligned} 1 &\mapsto a_1 \\ 2 &\mapsto a_2 \\ &\vdots \end{aligned}$$

Example

1. The set of prime numbers $p_1, p_2, \ldots$ can be enumerated, hence is countably infinite.
2. The elements of $\mathbb{R}$ cannot be enumerated one after the other by Cantor's diagonal argument.

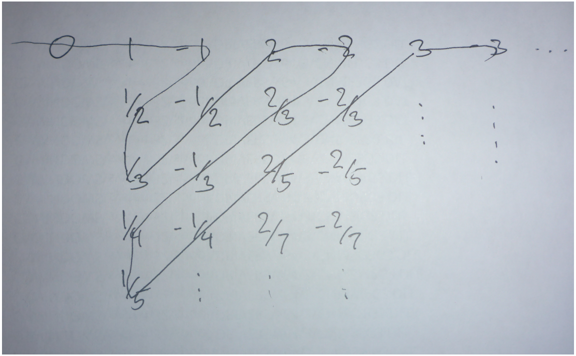
$\mathbb{Q}$ is countable

Theorem

$|\mathbb{Q}| = \aleph_0$

Proof.

Enumerate $\mathbb{Q}$



□

Similarly $\mathbb{N} \times \mathbb{N}, \mathbb{Z}^3, \ldots$ can be enumerated.

The Continuum Hypothesis

Recall $|\mathbb{N}| < |\mathbb{R}|$

Continuum Hypothesis (CH)

There is no set whose cardinality is strictly between $|\mathbb{N}|$ and $|\mathbb{R}|$.

- CH was proposed by Cantor 1878.
- CH can neither be disproved (Gödel 1940) nor proved (Cohen 1963) within the generally accepted foundations of Math, Zermelo-Fraenkel Set Theory (ZF).
- CH is **independent** from ZF; true or false depending on what additional axioms you accept to build your sets.
- Do you want to know more? Take a class like 'Math 4000 –Foundations of Math' this Spring.