# ZERMELO-FRAENKEL SET THEORY (ZF)

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Around 1900 Zermelo and Fraenkel proposed axioms to formalize the notion of sets and to avoid paradoxes like Russell's paradox. Since then they have established themselves as the most common foundation of mathematics. For now we describe them in a somewhat informal way.

(1) **Axiom of extensionality.** Sets $A$ and $B$ are equal if they have the same elements.
(2) **Axiom of regularity.** Every non-empty set $A$ has an element $B$ such that $A$ and $B$ are disjoint.
(3) **Axiom of specification.** For a set $A$ and a property $P$,
$$\{x \in A \ : \ x \text{ satisfies } P\}$$
is a set.
(4) **Axiom of pairing.** For any two sets $A$ and $B$, there exists the set $\{A, B\}$.
(5) **Axiom of unions.** For a set $I$ and sets $A_i$ for $i \in I$,
$$\bigcup_{i \in I} A_i := \{x \ : \ x \in A_i \text{ for some } i \in I\}$$
is a set.
(6) **Axiom of replacement.** For a set $A$ and a function $f$,
$$f(A) := \{f(x) \ : \ x \in A\}$$
is a set.
(7) **Axiom of infinity.** There exists a set that has infinitely many elements (e.g. $\mathbb{N}$).
(8) **Axiom of power set.** For every set $A$, there exists the set of all its subsets
$$P(A) := \{B \ : \ B \subseteq A\}.$$

## References

[1] Paul Halmos. Naive Set Theory. Springer Verlag, New York, 1974.

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