Math 2001 - Review Problems

(1) Show for all sets A, B:

$$P(A) \cup P(B) \subseteq P(A \cup B)$$

(2) (a) Negate the following statements:

(i) $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathbb{R}\colon |x| < \delta \Rightarrow x^2 < \varepsilon$

(ii) $x^y \in \mathbb{Q}$ iff $x \in \mathbb{Q}$ and $y \in \mathbb{N}$.

(b) Are $P \Rightarrow \sim Q$ and $\sim (P \wedge Q)$ logically equivalent?

(3) A word is a list of letters over the alphabet $A, \ldots, Z$.

How many 5-letter words are there that start with A or end with Z?

(4) Prove by contradiction: $x^2 - y^2 = 1$ has no solutions $x, y \in \mathbb{N}$. (Recall $0 \notin \mathbb{N}$ and consider a factorization of $x^2 - y^2$.)

(5) Show for any $n \in \mathbb{N}$ with $n \geq 3$:

$$\sum_{i=2}^{n-1} \binom{i}{2} = \binom{n}{3}.$$

(6) (a) The kernel of a function $f : A \to B$ is the relation

$$R = \{(x, y) \in A^2\ :\ f(x) = f(y)\}.$$

Show that R is an equivalence relation on A.

(b) For $x, y \in \mathbb{Z}_6$ let

$$x\,S\,y \text{ if } x^2 = y^2.$$

Then S is an equivalence relation on $\mathbb{Z}_6$. Describe the equivalence classes of S. How many are there?

(7) Solve $[9] \cdot [x] = [5]$ in $\mathbb{Z}_{43}$ using the extended Euclidean algorithm.

(8) (a) Check whether the function

$$f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, (x, y) \mapsto x + 2y,$$

is injective, surjective, bijective.

(b) Let $f\colon A \to B$ and $g\colon B \to C$. Prove that if $g \circ f$ is injective, then f is injective.