# Math 2001 - Assignment 8

Due October 23, 2015

(1) Read Section 5.3 in [1].

(2) Solve the following for $u, v \in \mathbb{Z}$:
   (a) $33u + 10v = -5$   (b) $44u + 10v = 5$

(3) Let $a, b, c \in \mathbb{Z}$ with $a, b$ not both 0. Show that

$$\exists u, v \in \mathbb{Z} \colon u \cdot a + v \cdot b = c \text{ iff } \gcd(a, b) | c.$$

Hint: There are 2 implications to show:
   (a) If $u \cdot a + v \cdot b = c$, then $\gcd(a, b)|c$.
   (b) If $\gcd(a, b)|c$, then there are $u, v \in \mathbb{Z}$ such that $u \cdot a + v \cdot b = c$.

(4) Two integers have the *same parity* if both are even or both are odd. Otherwise they have *opposite parity*.
   Let $a, b \in \mathbb{Z}$. Show that if $a + b$ is even, then $a, b$ have the same parity.
   Hint: Use a contrapositive proof.

(5) A natural number is *composite* if it is not prime.
   Show that for $n \in \mathbb{N}, n > 1$, every number in the sequence $n! + 2, n! + 3, \ldots, n! + n$ is composite.
   Hence one can produce arbitrary long intervals of natural numbers that contain no prime.

(6) Complete the following proof of **Euclid's Lemma:**
   Let $p$ be a prime, $a, b \in \mathbb{Z}$. If $p|ab$, then $p|a$ or $p|b$.
   *Proof:* Assume ______________ but $p \not| a$. We will show $p|b$.
   By Bezout's identity we have $u, v \in \mathbb{Z}$ such that

$$\underline{\hspace{6cm}} = \gcd(a, b).$$

Since $p$ is ______________ and $p \not| a$, we have $\gcd(a, p) =$_____.
Hence

$$ua + vp = \underline{\hspace{1.5cm}}.$$

Multiplying this equation by _____ yields

$$\underline{\hspace{6cm}} = b$$

Since $p|$______________ and $p|$______________, we have a multiple of $p$ on the left hand side of this equation. Thus __________.
$\square$

Please hand in this sheet of paper with your solution of 6.

[1] Richard Hammack. The Book of Proof. Creative Commons, 2nd edition, 2013.