Four (five) types of subuniverses and the complexity of the constraint satisfaction problem

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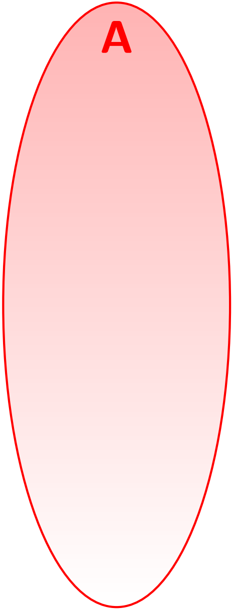
Joint Mathematics Meetings
AMS Special Session on Algebras and Algorithms

General approach



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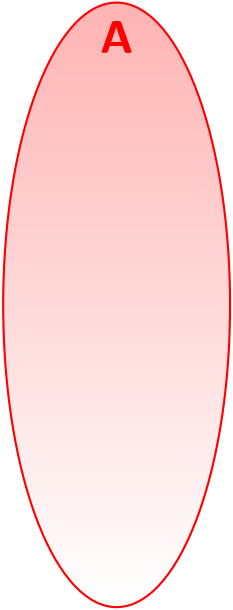

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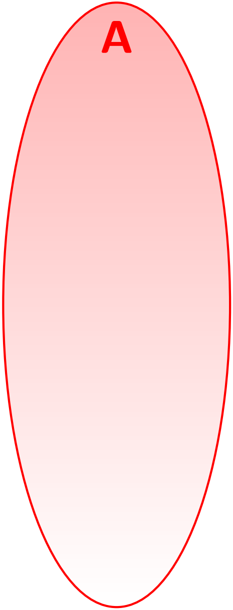

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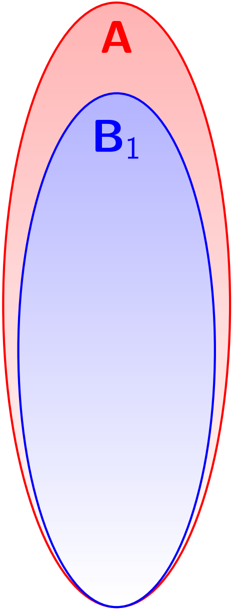

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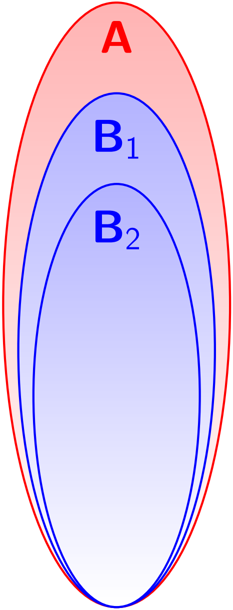

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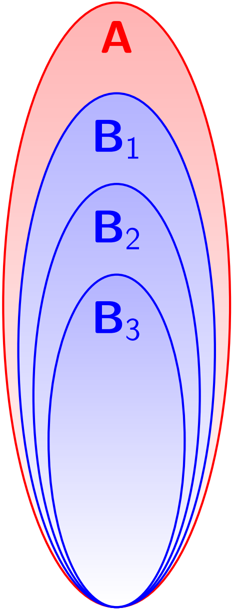

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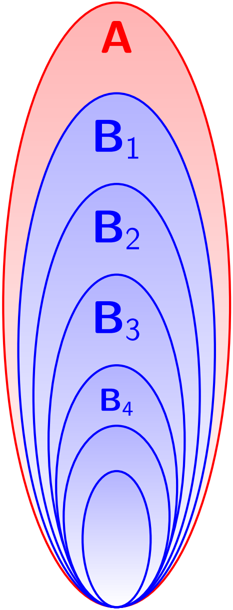

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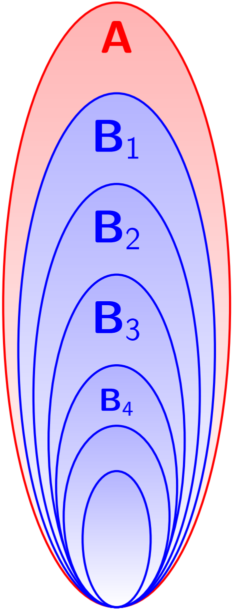

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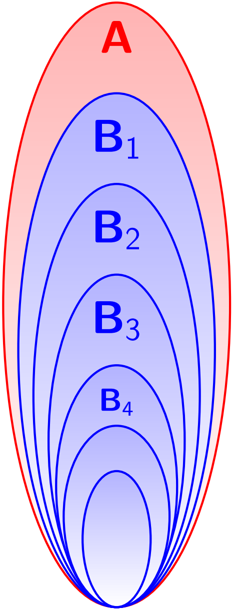

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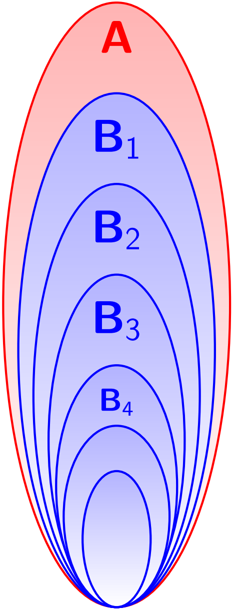

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2. When $\mathbf{B}_i$ is small enough derive a contradiction or required fact.

Three claims

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Claim 1 [M. Maróti and R. Mckenzie, 2008]

Every finite idempotent algebra $\mathbf{A}$

1. has a WNU* term operation, or
   * $w(y, x, \ldots, x) = w(x, y, x, \ldots, x) = \cdots = w(x, \ldots, x, y)$
2. has essentially unary $\mathbf{C} \in HSP(\mathbf{A})$.

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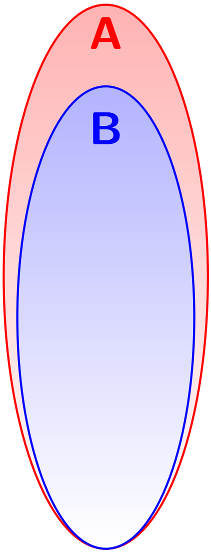
Claim 3 [M. Kozik. 2016]

Every cycle-consistent ($(2,3)$-consistent) CSP instance $\mathcal{I}$ over a constraint language with bounded width has a solution.

Absorbing subuniverse



B is an absorbing subuniverse if there exists a term operation such that

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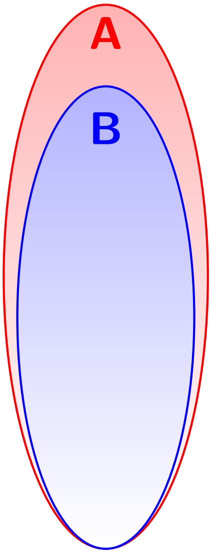

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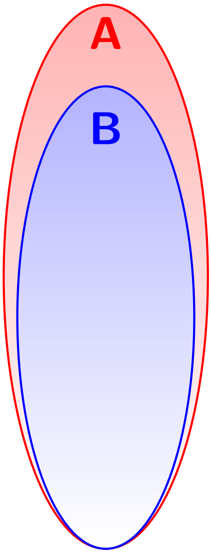

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Examples

1. $\{1\}$ is s binary absorbing subuniverse in $(\{0, 1\}, \vee)$.
2. $\{2, 3\}$ is a binary absorbing subuniverse in $(\{0, 1, 2, 3\}, max)$.
3. $\{2\}$ is an absorbing subuniverse in $(\{0, 1, 2, 3\}, majority)$.

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3. a linear subuniverse B, i.e a block of a congruence σ s.t. A/σ is an affine square-free algebra, i.e.
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5. a CBT (Cube Term Blocker) subuniverse B, i.e. $A^n \setminus (A \setminus B)^n \in \mathrm{Inv}(\mathbf{A})$ for every n.

Properties of strong subalgebras

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(Here $\mathcal{T} \in \{BA(t), C, L, PC\}$)

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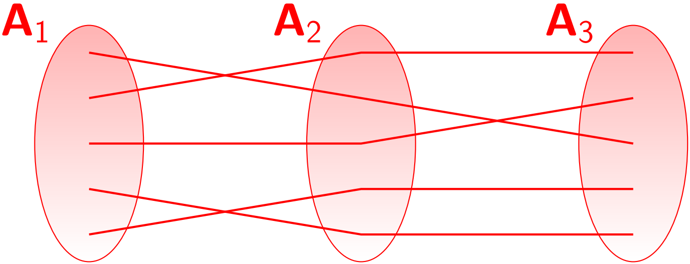

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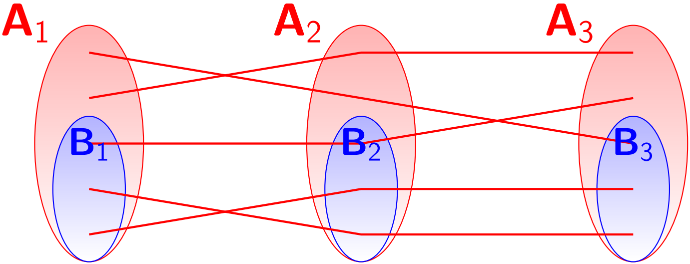

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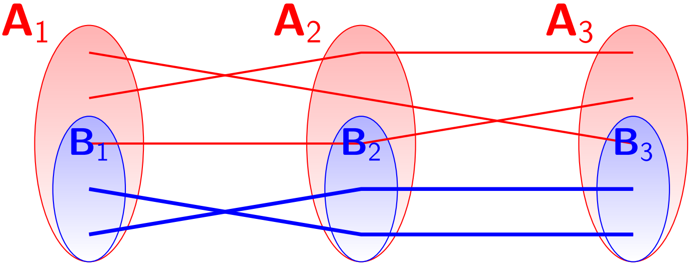

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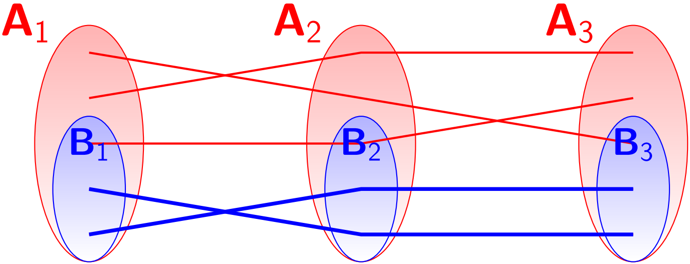

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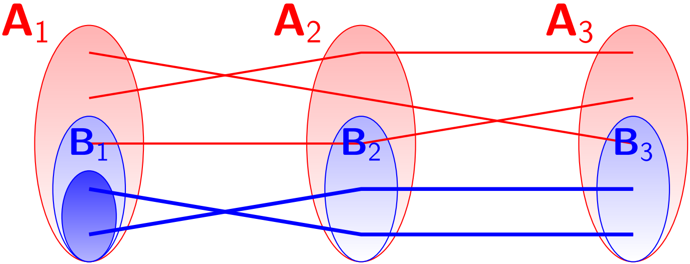

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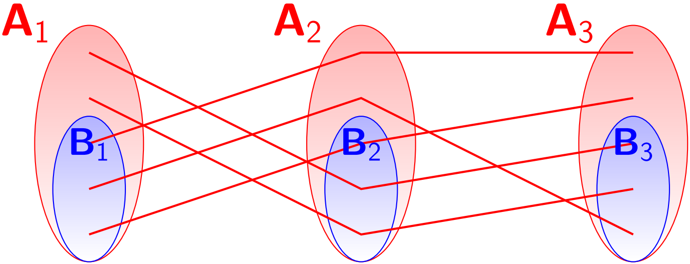

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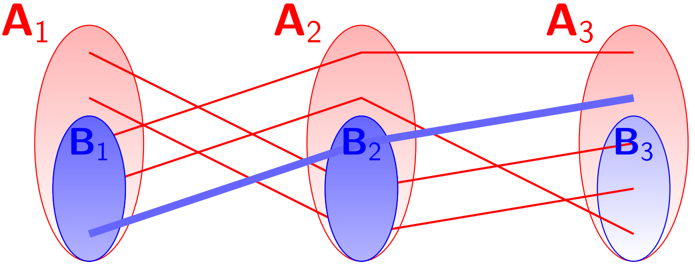

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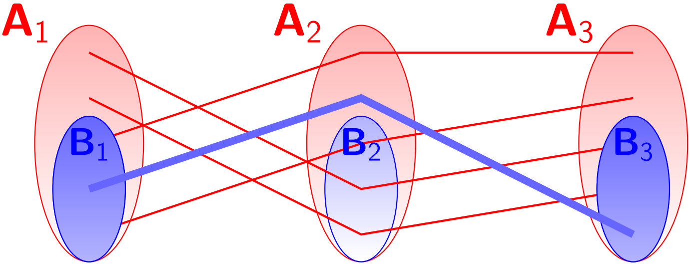

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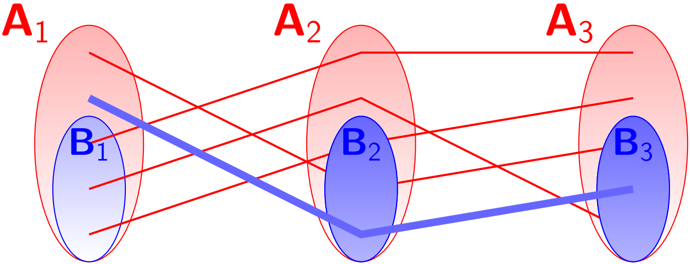

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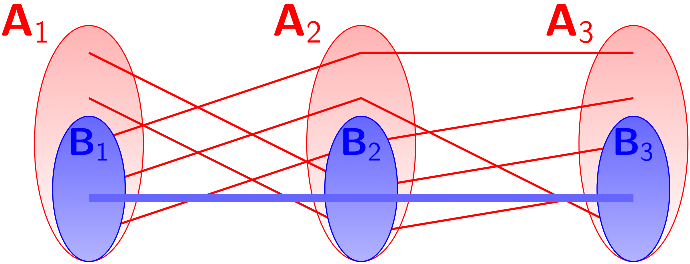

Properties of strong subalgebras

We write $\mathbf{B} \leq_{\mathcal{T}} \mathbf{A}$ if B is a subuniverse of $\mathbf{A}$ of type $\mathcal{T}$
(Here $\mathcal{T} \in \{BA(t), C, L, PC\}$)

Theorem

Suppose $R \leq_{sd} \mathbf{A}_1 \times \cdots \times \mathbf{A}_s$, $\mathbf{B}_i \leq_{\mathcal{T}} \mathbf{A}_i$ for every i. Then

1. $R \cap (B_1 \times \cdots \times B_n) \leq_{\mathcal{T}} R$
2. $\mathrm{proj}_1(R \cap (B_1 \times \cdots \times B_n)) \leq_{\mathcal{T}} \mathbf{A}_1$ *whenever* $\mathbf{A}_1$ *has no binary absorption or center, or* $\mathcal{T} \notin \{L, PC\}$
3. $R \cap (B_1 \times \cdots \times B_{i-1} \times A \times B_{i+1} \times \cdots \times B_n) \neq \varnothing$ *for every* i *implies* $R \cap (B_1 \times \cdots \times B_n) \neq \varnothing$ *whenever*
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Suppose $R \leq \mathbf{A}^p$ is a (nonempty) totally symmetric relation, where $p > |A|$ is a prime number. Then

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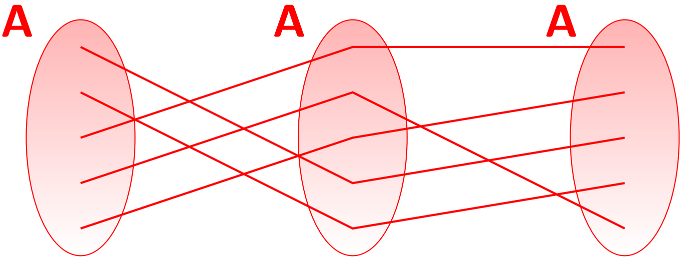

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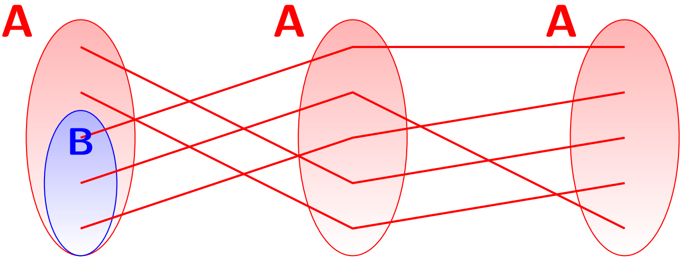

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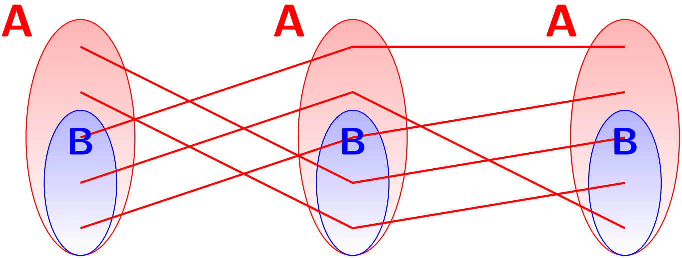

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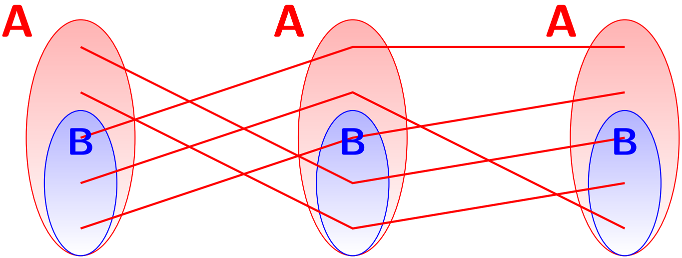

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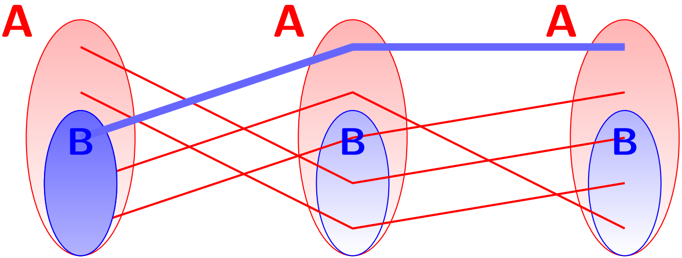

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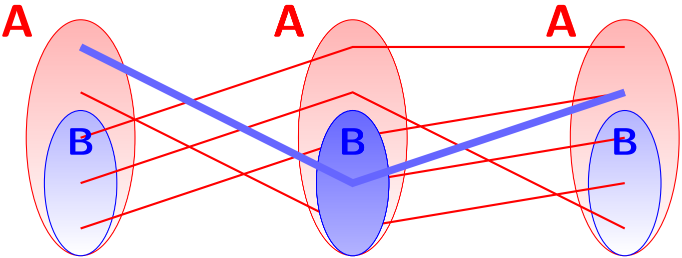

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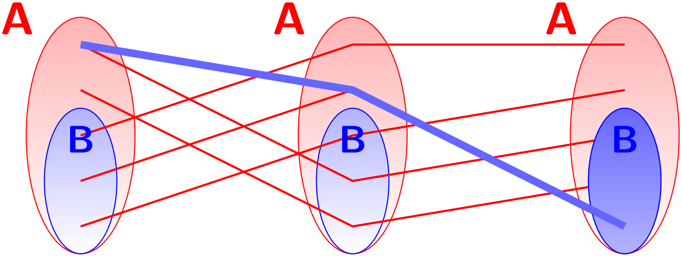

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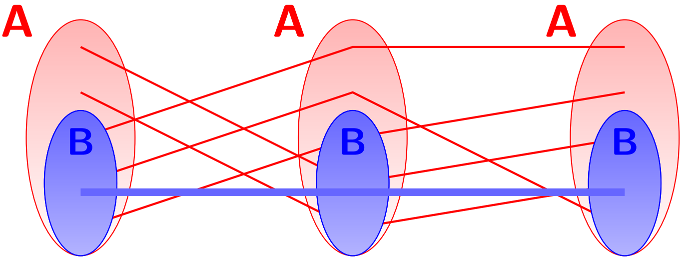

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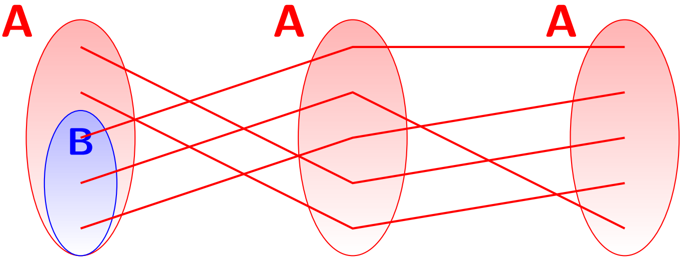

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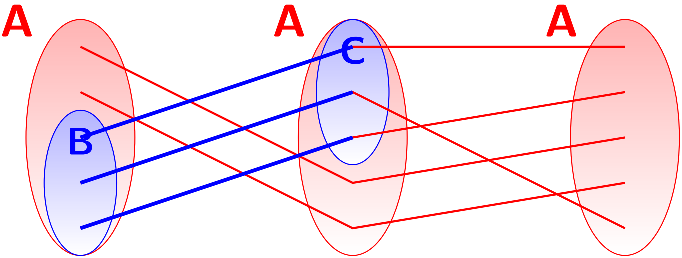

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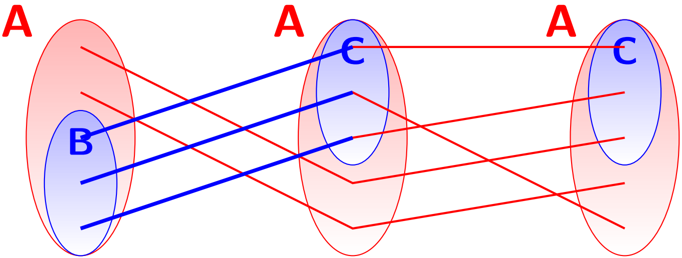

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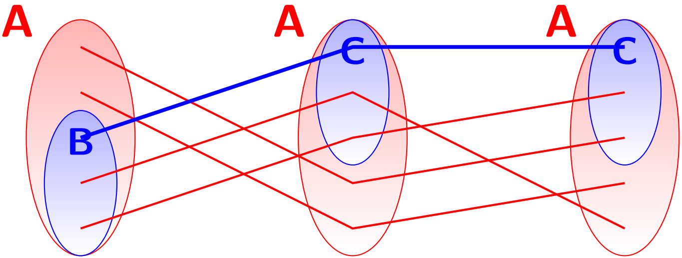

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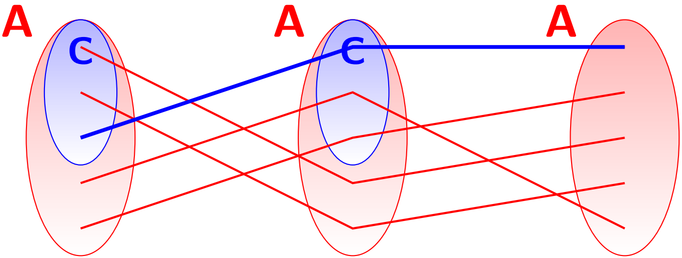

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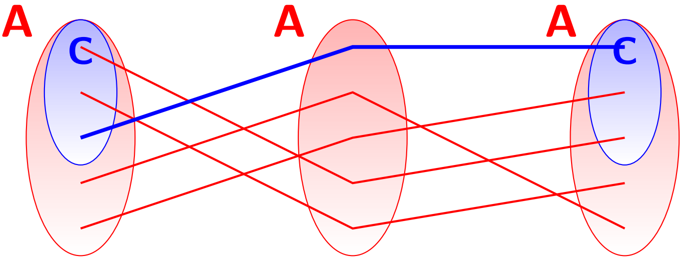

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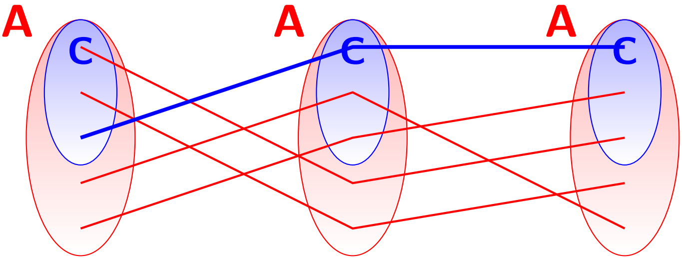

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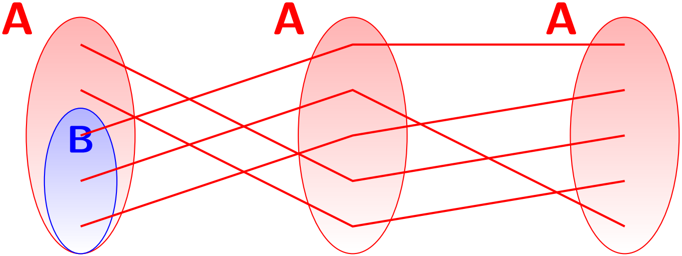

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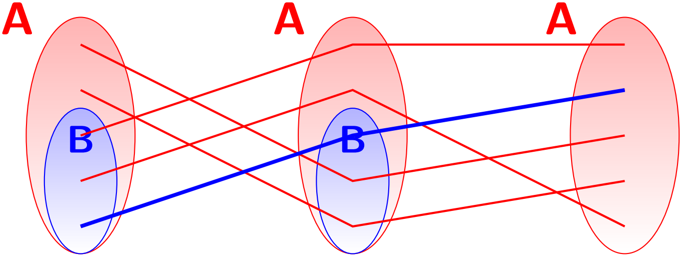

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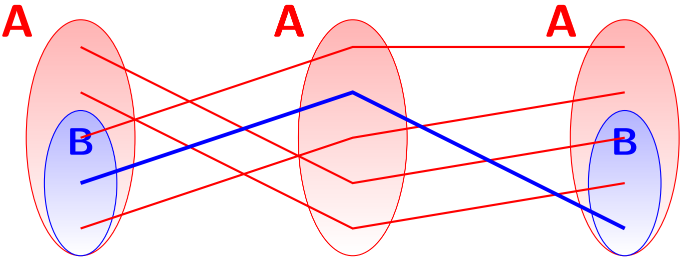

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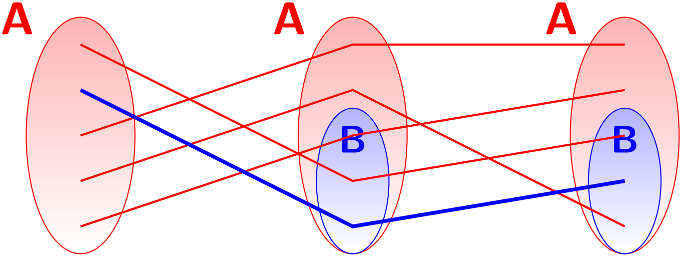

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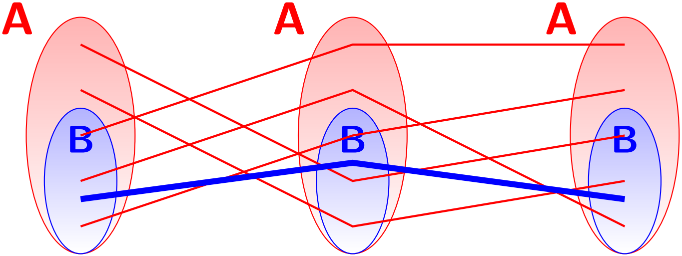

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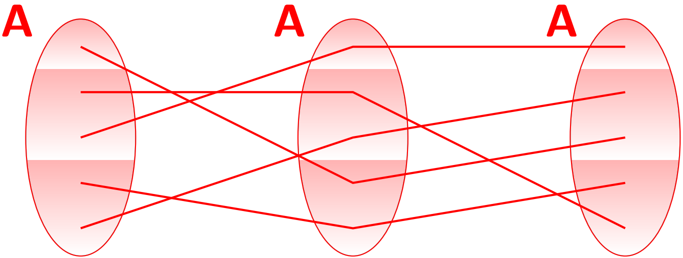

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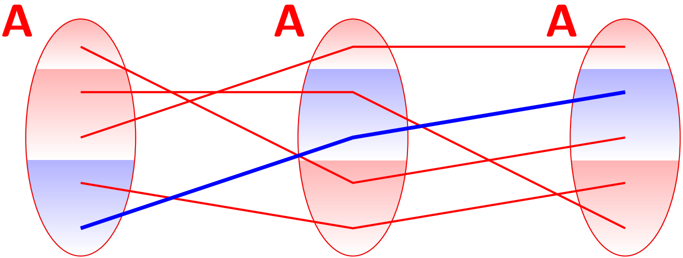

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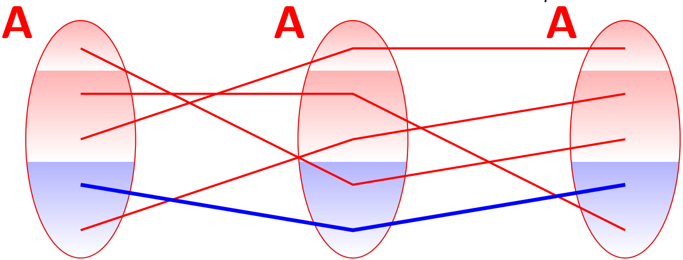

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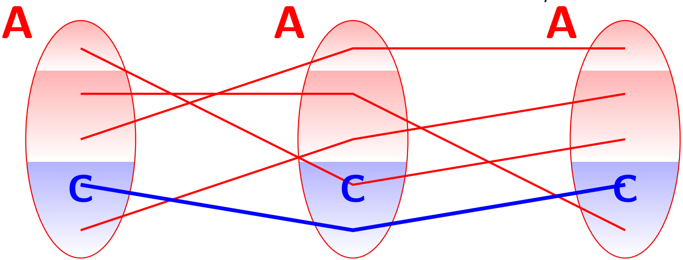

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Every finite idempotent algebra **A**

1. has a WNU* term operation, or
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Claim 1 [M. Maróti and R. Mckenzie, 2008]

Every finite idempotent algebra **A**

1. has a WNU* term operation, or
   * $w(y, x, \ldots, x) = w(x, y, x, \ldots, x) = \cdots = w(x, \ldots, x, y)$
2. has essentially unary $\mathbf{C} \in HSP(\mathbf{A})$.

Proof

Consider a free algebra over $\{x, y\}$ and generate a relation R from

$$\begin{pmatrix} y & x & \ldots & x \\ x & y & \ldots & x \\ \vdots & \vdots & \ddots & \vdots \\ x & x & \ldots & y \end{pmatrix}.$$

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Lemma 2

Suppose $\mathbf{A}$ is a finite idempotent algebra with bounded width, $R \leq \mathbf{A}^n$ is a (nonempty) totally symmetric relation, where $n \geq 3$. Then there exists $(a, a, \ldots, a) \in R$.

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Claim 2 [M. Kozik, A. Krokhin, M. Valeriote, R. Willard, 2015]

Every finite idempotent algebra **A** with bounded width has a WNU* term operation of every arity greater than two.

* $w(y, x, \ldots, x) = w(x, y, x, \ldots, x) = \cdots = w(x, \ldots, x, y)$

Constraint language

Let Γ be a set of (multi-sorted) relations.

Set of its polymorphism forms an algebra on every domain.

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CSP Instance

CSP instance over a constraint language Γ is

$$R_1(\dots) \wedge R_2(\dots) \wedge \dots \wedge R_s(\dots),$$

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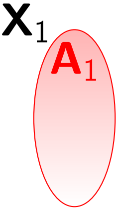
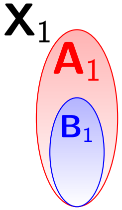
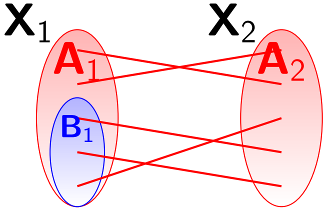
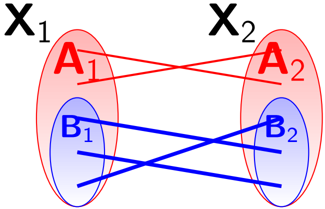
Claim 3 [M. Kozik. 2016]

Every cycle-consistent ((2,3)-consistent) CSP instance $\mathcal{I}$ over a constraint language with bounded width has a solution.

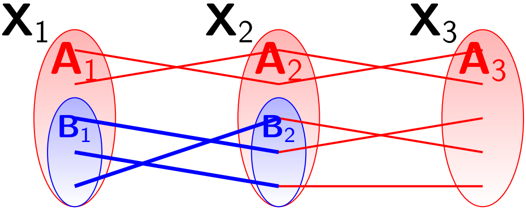

$\mathbf{X}_1$
$\mathbf{A}_1$


$\mathbf{X}_1$
$\mathbf{A}_1$
$\mathbf{B}_1$

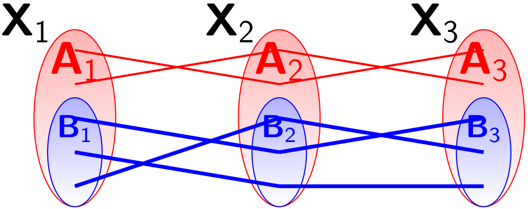

X1
X2
A1
A2
B1

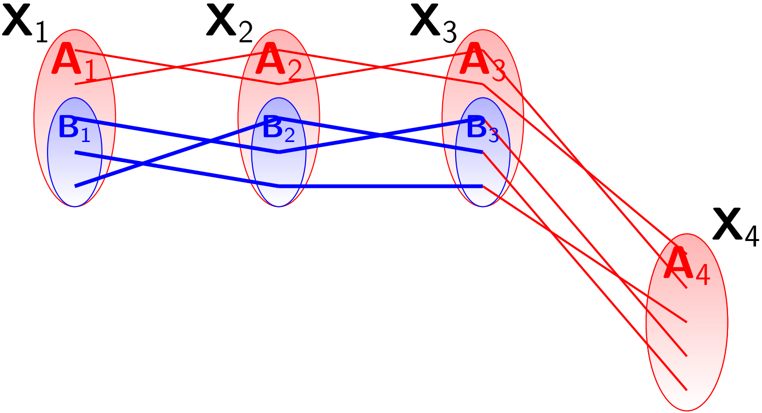


X_1
X_2
A_1
A_2
B_1
B_2

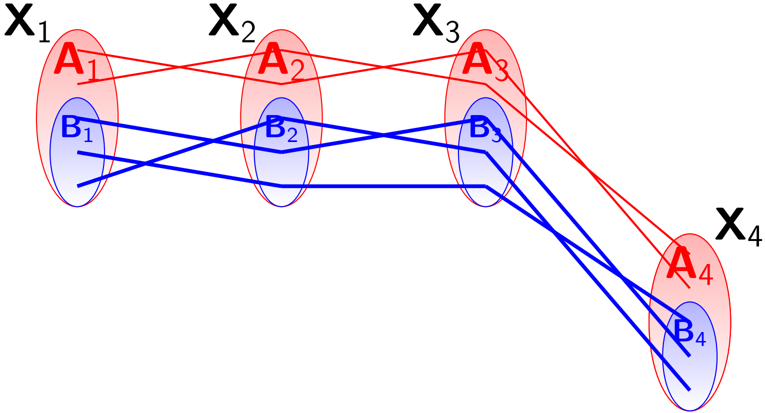


$\mathbf{X}_1$
$\mathbf{X}_2$
$\mathbf{X}_3$
$\mathbf{A}_1$
$\mathbf{A}_2$
$\mathbf{A}_3$
$\mathbf{B}_1$
$\mathbf{B}_2$

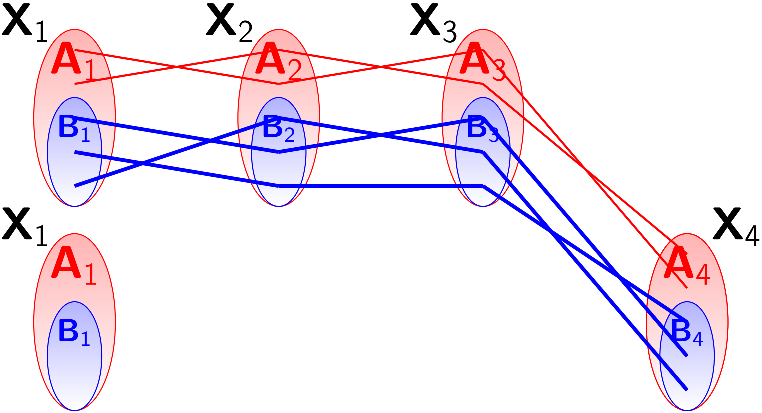

X1
X2
X3
A1
A2
A3
B1
B2
B3



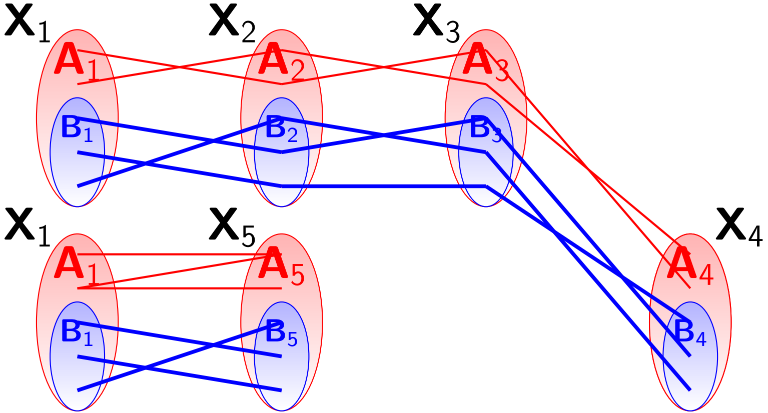
X1
X2
X3
X4
A1
A2
A3
A4
B1
B2
B3



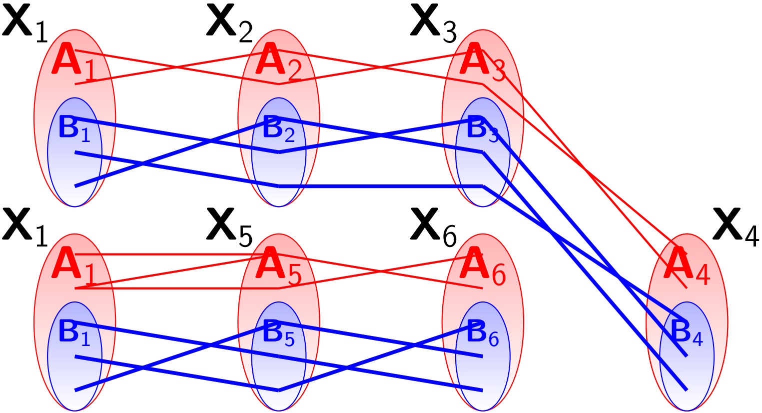
X_1
X_2
X_3
X_4
A_1
A_2
A_3
A_4
B_1
B_2
B_3
B_4



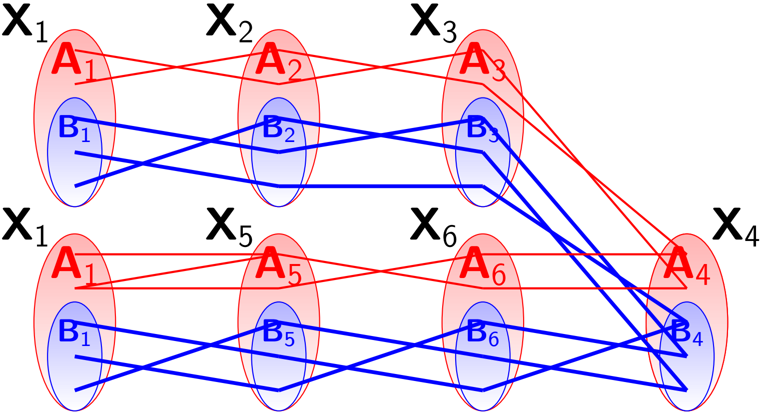
X1
A1
B1
X2
A2
B2
X3
A3
B3
X1
A1
B1
X4
A4
B4



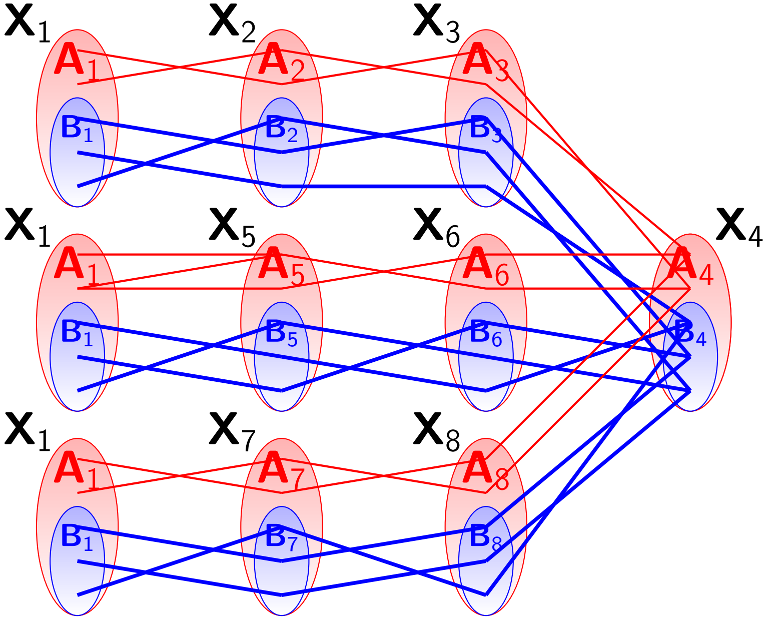
X_1
X_2
X_3
A_1
A_2
A_3
B_1
B_2
B_3
X_1
X_5
A_1
A_5
B_1
B_5
X_4
A_4
B_4



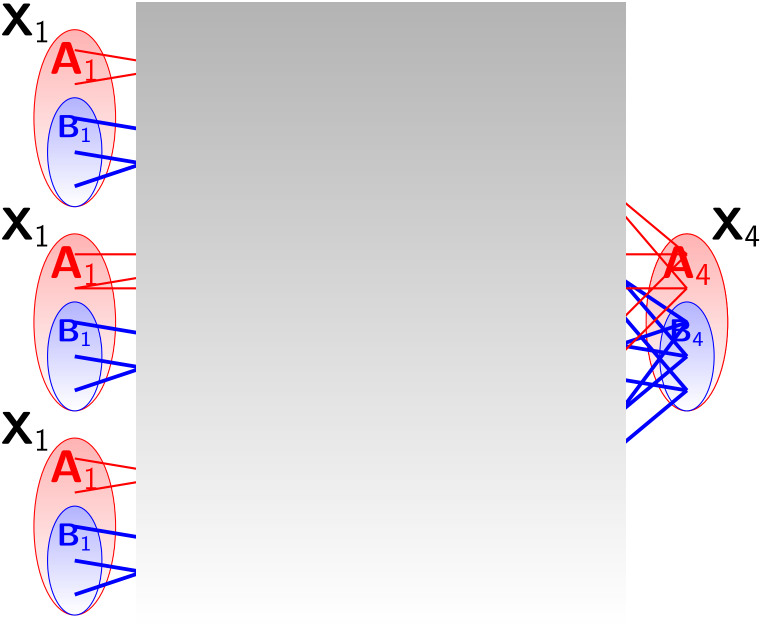
X1
X2
X3
A1
A2
A3
B1
B2
B3
X1
X5
X6
X4
A1
A5
A6
A4
B1
B5
B6
B4



X_1
X_2
X_3
A_1
A_2
A_3
B_1
B_2
B_3
X_1
X_5
X_6
X_4
A_1
A_5
A_6
A_4
B_1
B_5
B_6
B_4

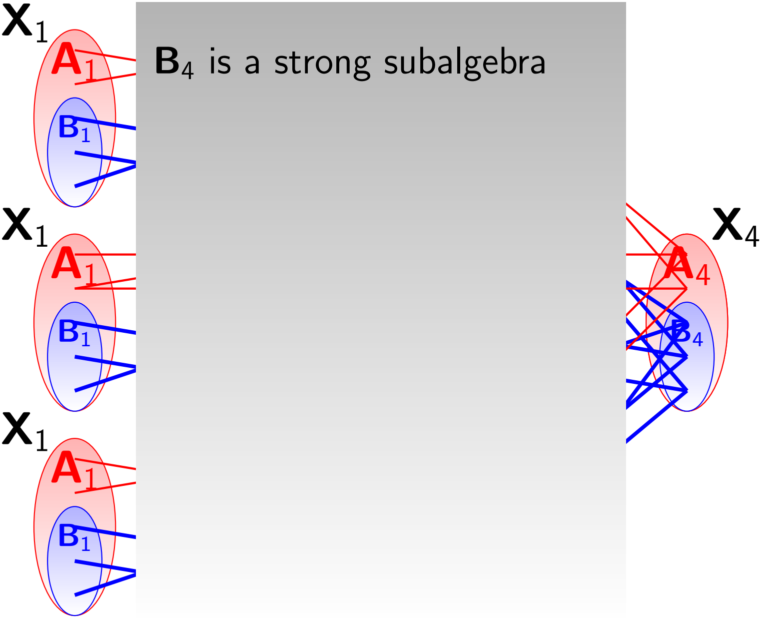


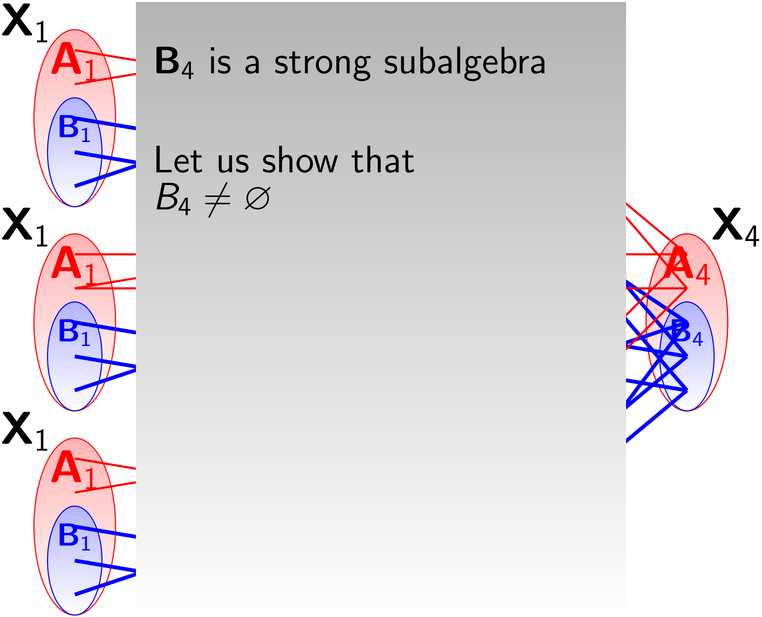
X_1
A_1
B_1
X_2
A_2
B_2
X_3
A_3
B_3
X_1
A_1
B_1
X_5
A_5
B_5
X_6
A_6
B_6
X_4
A_4
B_4
X_1
A_1
B_1
X_7
A_7
B_7
X_8
A_8
B_8



X1
A1
B1
X1
A1
B1
X1
A1
B1
X4
A4
B4

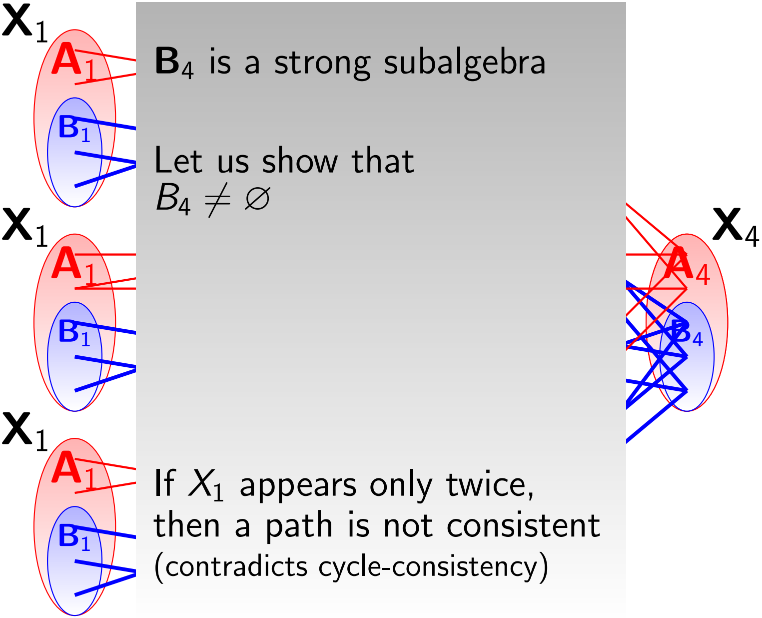
$\mathbf{B}_4$ is a strong subalgebra





$\mathbf{B}_4$ is a strong subalgebra

Let us show that
$B_4 \neq \varnothing$



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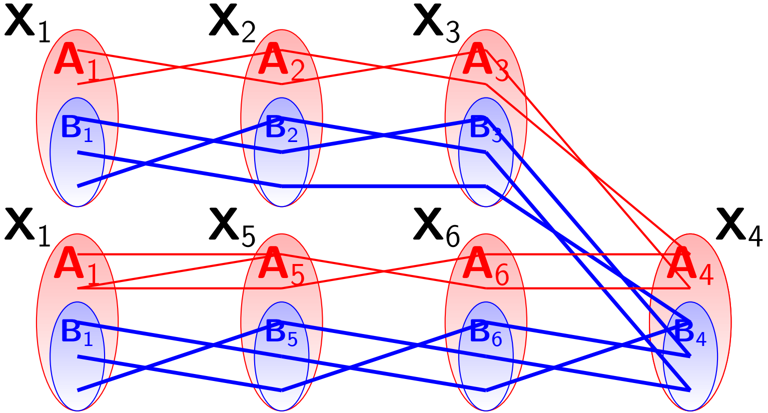
If X_1 appears only twice,
then a path is not consistent
(contradicts cycle-consistency)



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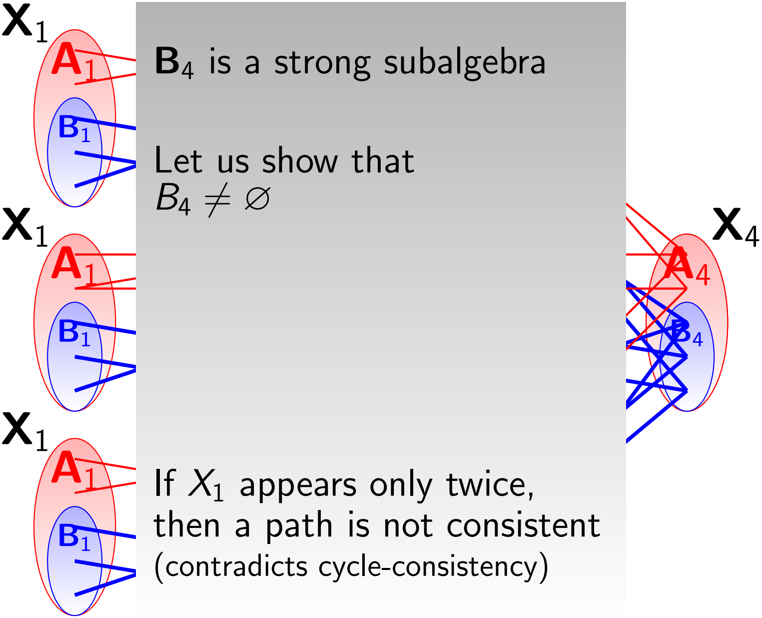

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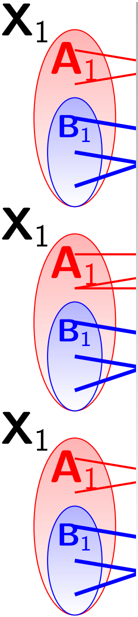

B_4 is a strong subalgebra

Let us show that
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If X_1 appears more than twice
it follows from the property

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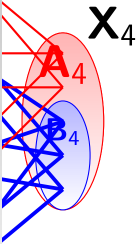
then a path is not consistent
(contradicts cycle-consistency)



Thank you for your attention