Finite algebras and regular tree languages

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joint work with Mikołaj Bojańczyk

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- A Σ-language L is regular if it is recognized by some finite algebra.

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$\mathbb{E} = \langle \{0, 1, 2\}, a^{\mathbb{E}}, b^{\mathbb{E}}, c^{\mathbb{E}} \rangle$,

$b^{\mathbb{E}}$	0	1	2
0	1	2	2
1	2	0	2
2	2	2	2

x	$a^{\mathbb{E}}(x)$
0	1
1	0
2	2

$c^{\mathbb{E}} = 0$

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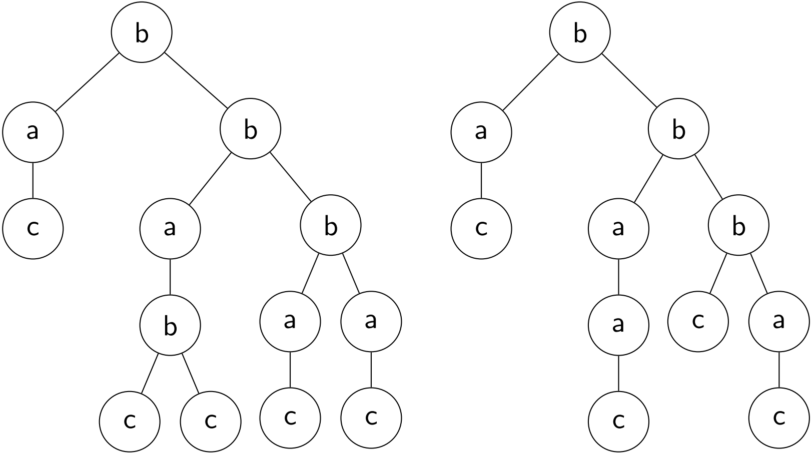

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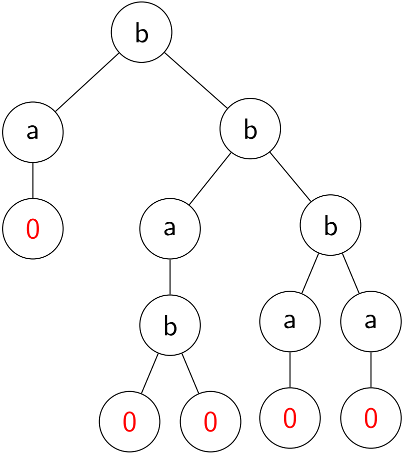

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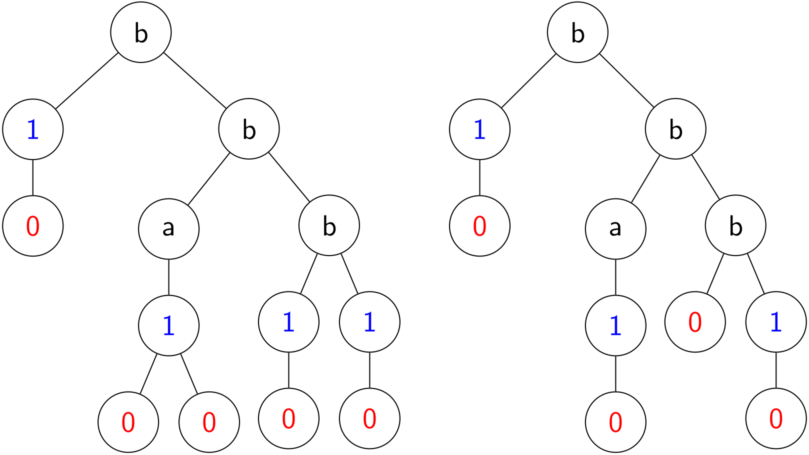

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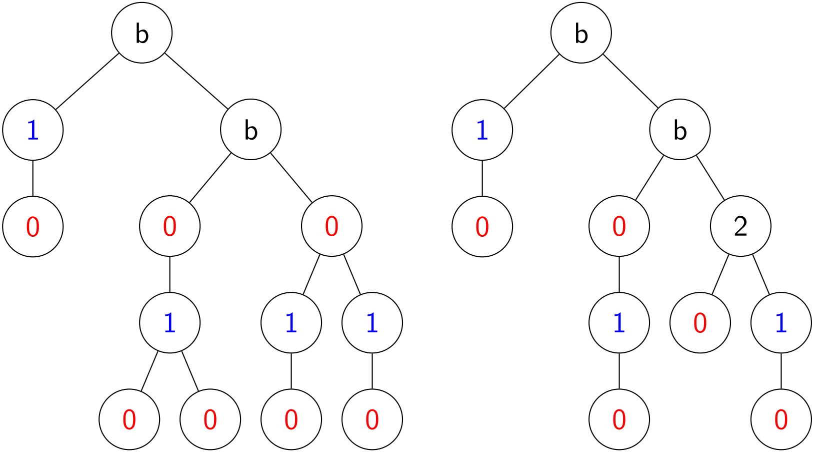

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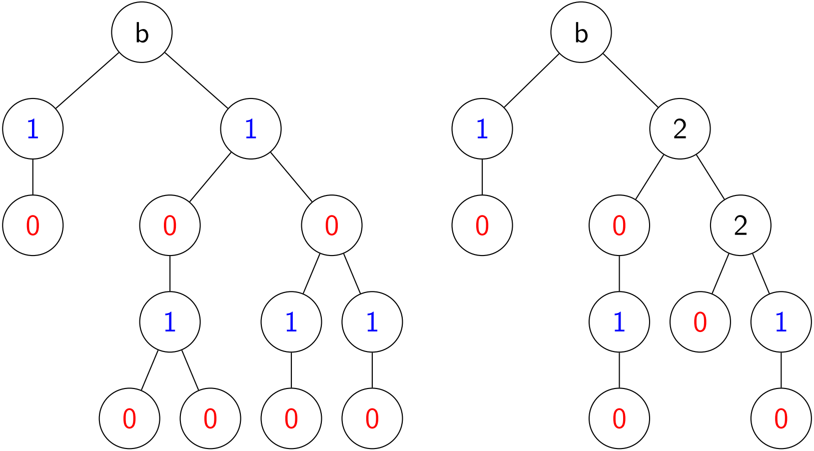

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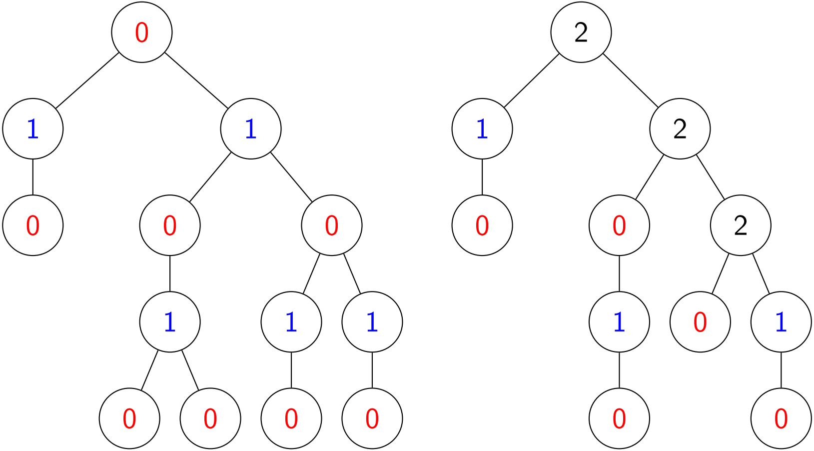

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Example

The first order sentence:

$$\exists x\,((\forall y \quad y \leq x) \wedge a(x)) \wedge \exists z(b(z))$$

defines the set of Σ-trees that have the symbol a as their root and in which b occurs at least once.

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Definability problem

The definability problem for a given fragment of MSO, such as first-order or chain logic, is: given a regular tree language, decide if it is definable by some formula of the logic.

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  Every maximal chain of t can be partitioned into two subchains X and Y that interleave such that

  - X contains the leaf of the branch and

  - Y contains the root.
- This tree language is not first order definable, since the property of being of even length cannot be defined in the first order language of Σ-trees.

Definability and the syntactic algebra

Syntactic Algebra

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Theorem (Schützenberger)

Let L be a regular language over an alphabet of rank 1. Then L is first-order definable if and only if there is some $n > 0$ *such that* $\mathbb{A}_L \models t^n(x) \approx t^{n+1}(x)$ *for all unary terms* t*.*

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Both problems are still open.

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Matrix powers

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Let $\mathbb{A}$ be an algebra with clone $\mathcal{C} = \mathrm{Clo}(\mathbb{A})$ and let $n > 0$. The *n-th matrix power of $\mathbb{A}$* is the algebra with universe A^n and whose basic operations are all functions $f(\bar{x}_1, \ldots, \bar{x}_k)$ of the form: for some choice of nk-ary functions $f_1, f_2, \ldots, f_n \in \mathcal{C}$,

$$(\bar{a}_1, \ldots, \bar{a}_k) \mapsto (f_1(\bar{a}_1, \ldots, \bar{a}_k), \ldots, f_n(\bar{a}_1, \ldots, \bar{a}_k))$$

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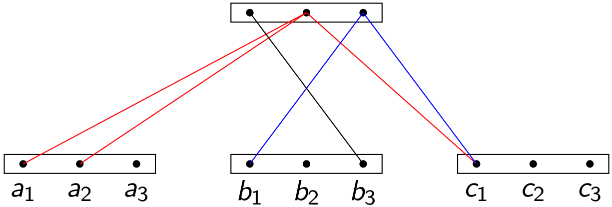
- *The matrix power construction is a generalization of the cartesian power of an algebra in the sense that* $\mathbb{A}^n$ *is a reduct of* $\mathbb{A}^{[n]}$.

Matrix Power Example

The function

$$((a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3)) \to (b_3, a_1 \vee a_2 \vee c_1, b_1 \vee c_1)$$

is in the clone of $\langle \{0, 1\}, \vee \rangle^{[3]}$.



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- For $\mathcal{K}$ a class of algebras, let $\mathbf{Wr}(\mathcal{K})$ denote the class of all wreath products of members from $\mathcal{K}$.

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- *So, the smallest class of algebras that contains* $\mathcal{K}$ *and is closed under division, matrix powers, and wreath products is* $\mathbf{DWrM}(\mathcal{K})$, *i.e.,* $\mathbf{Sim}(\mathcal{K}) = \mathbf{DWrM}(\mathcal{K})$.

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Remark

Using the work of VanderWerf, this problem reduces to the problem of determining if there is a procedure to decide if a given finite simple algebra of semilattice type is equal to a divisor of wreath products of matrix powers of $\langle\{0,1\},\vee\rangle$.

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- the class of finite solvable algebras is equal to $\mathbf{Sim}(\{\mathbb{Z}_p \mid p \text{ prime}\})$.
- If $\mathbb{A}$ is a finite algebra that admits the lattice or boolean type then $\mathbf{Sim}(\mathbb{A})$ is the class of all finite algebras.