Maltsev conditions invariant under permutation group actions

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Maltsev conditions

- Functional equations with a solution in **A**:
- An example:

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\begin{aligned}
p(x,x,y) &\approx p(y,y,y) \\
p(y,x,x) &\approx p(y,y,y) \\
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Why should we care?

- Fun with relations and algebras
- Maltsev conditions tell us about symmetries of admissible relations (cf. loop conditions)
- Good proving ground for algorithms
- UACalc
- PCSP = deciding Maltsev conditions (but: Maltsev conditions, not algebras, are the input)

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Local to global example

- If **A** is an algebra with a binary symmetric operation, then for any a, b there is q

$$\begin{pmatrix} q \\ q \end{pmatrix} \in \mathrm{Sg}_{\mathbf{A}^2} \left\{ \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} b \\ a \end{pmatrix} \right\}$$

- The converse holds for **A** finite idempotent
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Relational view

- **A** has idempotent local binary symmetric terms
- R is a relation containing

$$\begin{pmatrix} a \\ b \\ a' \\ b' \end{pmatrix} \begin{pmatrix} b \\ a \\ b' \\ a' \end{pmatrix}$$

- Then R also contains

$$\begin{pmatrix} q \\ q \\ t(a',b') \\ t(b',a') \end{pmatrix}, \begin{pmatrix} q \\ q \\ t(b',a') \\ t(a',b') \end{pmatrix}, \begin{pmatrix} q \\ q \\ r \\ r \end{pmatrix}$$

- Keep going like this. . .

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The fall of local-to-global

- Minority operation

$$n(x,x,y) \approx n(x,y,x) \approx n(y,x,x) \approx y$$

- For any $n \geq 2$ there is an idempotent algebra of size $4n$ with local minorities, but no global minority [K, Opršal, Valeriote, Zhuk, to appear in Canadian Mathematical Bulletin]
- More examples?

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G-terms

- G ... permutation group on $[n]$
- Equations $t(x_1, \dots, x_n) \approx t(x_{g(1)}, \dots, x_{g(n)})$ for all $g \in G$
- How about we study efficiency of deciding if an idempotent algebra has a G-term for a fixed G? [suggested by Matt Valeriote]
- Complexity depends on the permutation group, not abstract group

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Local to global works if...

- G has a one element orbit (trivial G-term)
- G acts on itself by left/right translations
- n is even and G acts as the dihedral group
- To be continued...

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- $G = S_n$ for $n \geq 3$
- G has no fixed points, but there is a $g \in G$ with k orbits of the same size m and $n = km + 1$
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Counterexample construction

- G has no fixed points, but there is a $g \in G$ with one orbit of size $n-1$ and one fixed point
- Pick $A = \{0,1\} \times [n] \cup \mathbb{Z}_{n-1}$
- Two basic n-ary operations t_0, t_1.
- t_i is a G-term outside of $\{i\} \times [n]$
- t_i's are symmetric affine on $\mathbb{Z}_{n-1}$
- Usually t_i's map $\{0,1\} \times [n]$ to $\mathbb{Z}_{n-1}$
- t_i on $(\{i\} \times [n])^n$ counts how many times g was applied

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Where to go from here

- Classify G-terms for all permutation groups
- We need another algorithm for deciding linear Maltsev conditions
- More assumptions on the algebras? Assuming 2-nilpotence did not help me for S_3-terms
- Uniform subpower membership problem algorithms?
- Guess: There is a hard G-term condition out there...
- ...but S_3-terms are too small to be hard

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