Varieties of Semilattice Sums

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Maltsev Products

Let $\mathcal{V}$ and $\mathcal{W}$ be quasivarieties. The *Maltsev product of* $\mathcal{V}$ *and* $\mathcal{W}$ is the class

$$\begin{aligned}\mathcal{V} \circ \mathcal{W} = \{\, \mathbf{A} : (\exists \varrho \in \mathrm{Con}(\mathbf{A}))\ \mathbf{A}/\varrho \in \mathcal{W} \text{ and} \\ (\forall a \in A)\ a/\varrho \in \mathrm{Sub}(\mathbf{A}) \implies a/\varrho \in \mathcal{V} \,\}\,.\end{aligned}$$

Under modest conditions, $\mathcal{V} \circ \mathcal{W}$ will be a quasivariety:

- Finite similarity type *or*
- $\mathcal{W}$ is idempotent.

If $\mathcal{V}$ and $\mathcal{W}$ are varieties, will $\mathcal{V} \circ \mathcal{W}$ ever be a variety?

Never. Well, hardly ever.

Semilattice Sums

Assume our similarity type has
–no nullary operations *and*
–at least one nonunary operation.
Then there is a unique variety $\mathcal{S}$ that is term-equivalent to the variety of semilattices.

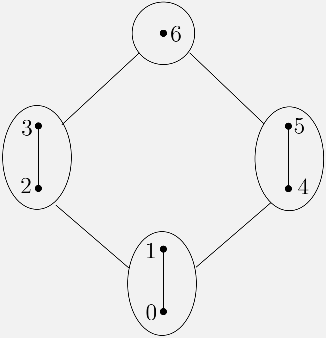
A member of $\mathcal{V} \circ \mathcal{S}$ is called a *semilattice sum of* $\mathcal{V}$*-algebras.*

Note that since $\mathcal{S}$ is idempotent, $\mathcal{V} \circ \mathcal{S}$ is a quasivariety. Also, $\mathbf{A} \in \mathcal{V} \circ \mathcal{S}$ implies every congruence class is a subalgebra.

Example

Assume our similarity type consists of a single binary operation symbol.

A	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0
2	0	0	2	2	0	0	2
3	0	0	2	3	0	1	2
4	0	0	0	0	4	4	4
5	0	0	0	0	4	5	4
6	0	0	2	3	4	5	6



$\varrho = |01|23|45|6|$

So **A** $\in \mathcal{S} \circ \mathcal{S}$. But $3 \cdot 5 \neq 5 \cdot 3$ so **A** $\notin \mathcal{S}$.

Regular Identities

An identity is *regular* if it has the same variables on each side of the equality

regular	**irregular**
$x \cdot x \approx x$	$x \cdot y \approx x$
$x \cdot y \approx y \cdot x$	$y \cdot (x \cdot y) \approx y \cdot (z \cdot y)$
$x \cdot (y \cdot z) \approx (x \cdot y) \cdot z$	$p(x, y, y) \approx x$
$x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$	$x \wedge (x \vee y) \approx x$

An identity of the form $t(x, y) \approx x$ is called *strongly irregular.*

Theorem: $\mathrm{Id}(\mathcal{S}) = \{$all regular identities$\}$.

Let σ be an identity

$$f(y_1, \ldots, y_n) \approx g(y_1, \ldots, y_n).$$

Define σ^* to be the *set* of all identities of the form

$$f\big(r_1(\mathbf{x}), \ldots, r_n(\mathbf{x})\big) \approx g\big(r_1(\mathbf{x}), \ldots, r_n(\mathbf{x})\big)$$

in which $m > 0$, $\mathbf{x} = x_1, \ldots, x_m$, and each r_i is a term on *exactly* m variables.

Straightforward to check:

$$\mathcal{V} \models \sigma \implies \mathcal{V} \circ \mathcal{S} \models \sigma^*.$$

Corollary

$\mathcal{V}$ a variety with equational base Σ. Let $\Sigma^* = \bigcup_{\sigma \in \Sigma} \sigma^*$ and $\mathcal{V}^* = \mathrm{Mod}(\Sigma^*)$. Then $\mathcal{V} \circ \mathcal{S} \subseteq \mathcal{V}^*$.

Main Theorem

Let $\mathcal{V}$ be any variety.

1. $\mathbf{H}(\mathcal{V} \circ \mathcal{S}) = \mathcal{V}^*$
2. If $\mathcal{V}$ is strongly irregular then $\mathcal{V} \circ \mathcal{S} = \mathcal{V}^*$.

What about regular varieties?

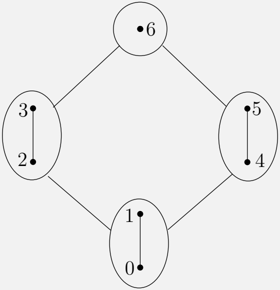
Counterexample

Let $\mathcal{C}$ be the variety of commutative binars. Consider

$$q : \quad (zx \approx x \;\&\; zy \approx y \;\&\; xz \approx yz \rightarrow xy \approx yx).$$

Easy to check: $\mathcal{C} \circ \mathcal{S} \models q$.

Recall our example: **A** =



$\mathbf{A} \in \mathcal{S} \circ \mathcal{S} \subseteq \mathcal{C} \circ S$. So $\mathbf{A} \models q$.

Let $\theta = \mathrm{Cg}(2, 4) = |024|1|3|5|6|$. Then $\mathbf{A}/\theta \not\models q$ with $x = 3/\theta$, $y = 5/\theta$, $z = 6/\theta$. Thus $\mathbf{A}/\theta \notin \mathcal{C} \circ \mathcal{S}$.

So $\mathcal{C} \circ \mathcal{S}$ not closed under homomorphic images

Presumably there is nothing special about commutativity.

On the other hand, let $\mathcal{A}$ be the variety of all algebras.
(Note: $\mathcal{A}$ is regular)
Then $\mathcal{A} \circ \mathcal{S} = \mathcal{A}$, which is a variety.

Conjecture: Let $\mathcal{V}$ be a proper, regular variety. Then $\mathcal{V} \circ \mathcal{S}$ is not closed under homs.

Key Ideas in the Proof

For $\mathbf{H}(\mathcal{V} \circ \mathcal{S}) = \mathcal{V}^*$

Let $\mathcal{W}$ be regular, $\mathbf{F} = \mathbf{F}_{\mathcal{W}}(X)$, $a \in F$.
$a = s^{\mathbf{F}}(x_1, \ldots, x_n)$ for some term s and $x_1, \ldots, x_n \in X$.
Regularity $\implies$ $\mathrm{var}(a) = \{x_1, \ldots, x_n\}$ is independent of s.

$\mathrm{var}\colon \mathbf{F} \to \mathrm{Sb}_{\omega}(X) - \{\varnothing\} \cong \mathbf{F}_{\mathcal{S}}(X)$
is a surjective homomorphism
$\therefore \quad \varrho = \ker(\mathrm{var})$.

Let $\mathcal{V}$ be arbitrary, $\mathcal{W} = \mathcal{V}^*$. Then $\mathcal{W}$ is regular.
From above, ϱ on $\mathbf{F}_{\mathcal{W}}(X)$ is known.

Identities of $\mathcal{V}^* \implies a/\varrho \in \mathcal{V},$ for all $a \in F$
$\therefore \mathbf{F}_{\mathcal{W}}(X) \in \mathcal{V} \circ \mathcal{S}$
$\therefore \mathbf{H}(\mathcal{V} \circ S) = \mathcal{V}^*$

Remark: With a suitable change to the definition of σ^*, the result holds with $\mathcal{S}$ replaced by an arbitrary idempotent variety.

For $\mathcal{V} \models t(x,y) \approx x \implies \mathcal{V} \circ \mathcal{S} = \mathcal{V}^*$

Assume $\mathcal{V} \models t(x,y) \approx x$.
$\mathbf{F} = \mathbf{F}_{\mathcal{V}^*}(X)$. So $\mathbf{F} \in \mathcal{V} \circ \mathcal{S}$.
Let $\theta \in \mathrm{Con}(\mathbf{F})$. Want $\mathbf{F}/\theta \in \mathcal{V} \circ \mathcal{S}$.
Equivalently, need $a \in F \implies a/(\theta \vee \varrho) \in \mathcal{V}$

strong irregularity $\implies \theta \vee \varrho = \theta \circ \varrho \circ \theta$.
Therefore each $(\theta \vee \varrho)$-class satisfies $t(x,y) \approx x$.
$\therefore \mathrm{Id}(\mathcal{V})^*$ imply each $(\theta \vee \varrho)$-class lies in $\mathcal{V}$.

The End