Math 3130 - Assignment 13

Due April 22, 2016

(109) [1, Section 6.1] Give 3 vectors of length 1 in $\mathbb{R}^3$ that are orthogonal to $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$.

Solution:
First we find 3 vectors orthogonal to $\mathbf{u}$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

Then we normalize these vectors so that the length of each vector is 1:

$$\mathbf{w}_1 = \frac{1}{||\mathbf{v}_1||}\mathbf{v}_1 = \frac{1}{\sqrt{17}} \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{w}_3 = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

□

(110) [1, Section 6.2] Which of the following are orthonormal sets?

$$A = \{\begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}, \begin{bmatrix} 0.8 \\ -0.6 \end{bmatrix}\}, \qquad B = \{\frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{18}}\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}\}$$

Solution:
Both sets are orthogonal since the dot product of distinct vectors is 0. In addition each vector has length 1. Thus A and B both are orthonormal. □

(111) [1, Section 6.2] Let W be the subspace of $\mathbb{R}^3$ with orthormal basis $B = (\frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix})$.

Compute the coordinates $[\mathbf{x}]_B$ for $\mathbf{x} = \begin{bmatrix} 7 \\ 4 \\ 4 \end{bmatrix}$.

Solution:
Let $(\mathbf{b}_1, \mathbf{b}_2) = B$. Then $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$ with $c_i = \frac{\mathbf{x}\cdot\mathbf{b}_i}{\mathbf{b}_i\cdot\mathbf{b}_i}$. We obtain $c_1 = 6$, $c_2 = \frac{15}{\sqrt{5}} = 3\sqrt{5}$, and $[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3\sqrt{5} \end{bmatrix}$. □

(112) [1, 14, Section 6.2] Write $\mathbf{x} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ as a sum of a vector in Span$\{\mathbf{u}\}$ and a vector in Span$\{\mathbf{u}\}^\perp$ for $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$.

Solution:

$$\operatorname{proj}_{\mathbf{u}} \mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \frac{2}{5} \begin{bmatrix} 7 \\ 1 \end{bmatrix}.$$

$$\mathbf{x} = \underbrace{\operatorname{proj}_{\mathbf{u}} \mathbf{x}}_{\in \operatorname{Span}\{\mathbf{u}\}} + \underbrace{(\mathbf{x} - \operatorname{proj}_{\mathbf{u}} \mathbf{x})}_{\perp \operatorname{Span}\{\mathbf{u}\}} = \frac{2}{5} \begin{bmatrix} 7 \\ 1 \end{bmatrix} + \frac{4}{5} \begin{bmatrix} -1 \\ 7 \end{bmatrix}.$$

□

(113) [1, 6, Section 6.3] Check that $\mathbf{u}_1 = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ form an orthogonal set and compute the orthogonal projection of $\mathbf{x} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$ onto $\operatorname{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$.
What is the distance from $\mathbf{x}$ to $\operatorname{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$?
Solution:
The set is orthogonal since $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$. The projection is given by

$$\operatorname{proj}_{\operatorname{Span}\{\mathbf{u}_1,\mathbf{u}_2\}} \mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{x} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 = \frac{-27}{18} \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} + \frac{5}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix} = \mathbf{x}.$$

The distance is

$$||\mathbf{x} - \operatorname{proj}_{\operatorname{Span}\{\mathbf{u}_1,\mathbf{u}_2\}} \mathbf{x}|| = ||\mathbf{0}|| = 0.$$

The fact that $\mathbf{x}$ is equal to its projection means that $\mathbf{x}$ is in $\operatorname{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Thus the distance is 0. □

(114) [1, 8, Section 6.3] Find the closest point to $\mathbf{x}$ in $\operatorname{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$:

$$\mathbf{x} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

Solution:
The closest point to $\mathbf{x}$ is

$$\operatorname{proj}_{\operatorname{Span}\{\mathbf{u}_1,\mathbf{u}_2\}} \mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{x} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 = \frac{6}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{7}{14} \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix}.$$

□

(115) [1, 2, Section 6.4] Use the Gram-Schmidt algorithm to find orthonormal bases for the following subspaces:

$$U = \operatorname{Span}\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} \}, \qquad W = \operatorname{Span}\{ \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix} \}$$

Solution:
Subspace U: Gram-Schmidt produces an orthogonal set:

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} - \frac{5}{5}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}$$

Normalization produces an orthonormal basis

$$(\frac{1}{\sqrt{5}}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{105}}\begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}).$$

Subspace W: Gram-Schmidt produces an orthogonal set:

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix} - \frac{-18}{9}\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We obtain $\mathbf{0}$ because the input was not linearly independent. We remove the zero vector and normalize to obtain an orthonormal basis

$$(\frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}).$$

□

(116) [1, Section 6.4] Use the Gram-Schmidt process to transform the vectors in an orthonormal set.

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 3 \\ -3 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 3 \\ -3 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ -2 \end{bmatrix},$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} - \frac{-2}{16}\begin{bmatrix} -2 \\ 2 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Normalization yields an orthonormal basis

$$(\frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}).$$

□

(117) [1, Section 6.2–6.4] True or false. Explain your answers.

(a) Every orthogonal set is also orthonormal.
(b) Not every orthonormal set in $\mathbb{R}^n$ is linearly independent.
(c) For each $\mathbf{x}$ and each subspace W, the vector $\mathbf{x} - \text{proj}_W(\mathbf{x})$ is orthogonal to W.
(d) If a vector is both in a subspace W and in $W^\perp$, then it must be the zero vector.
(e) Multiplying the vectors in an orthogonal basis by non-zero scalars yields again an orthogonal basis.

Solution:

(a) **False**. The set $(\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix})$ is orthogonal but not orthonormal.
(b) **False**. Every orthonormal set is an orthogonal set without the zero vector. Thus it is linearly independent by Theorem 6.4.
(c) **True** by Theorem 6.8.
(d) **True**. If $\mathbf{x}$ is in W and $W^\perp$, then $\mathbf{x} \cdot \mathbf{x} = 0$ and thus $\mathbf{x} = \mathbf{0}$.
(e) **True** since the new set is also orthogonal and spans the same vector space.

□

References

[1] David C. Lay. Linear Algebra and Its Applications. Addison-Wesley, 4th edition, 2012.