Math 3130 - Assignment 12

Due April 15, 2016

(100) [1, Section 6.1] Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Show that $(\mathbf{u}+\mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$.
Solution:

$$\begin{aligned}
(\mathbf{u}+\mathbf{v}) \cdot \mathbf{w} &= (\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}) \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \\
&= (u_1+v_1)w_1 + \cdots + (u_n+v_n)w_n \\
&= u_1 w_1 \cdots + u_n w_n + v_1 w_1 + \cdots v_n w_n \\
&= \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \\
&= \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}.
\end{aligned}$$

□

(101) [1, Section 6.1] Let $\mathbf{u} \in \mathbb{R}^n$. Show that
(a) $\mathbf{u} \cdot \mathbf{u} \geq 0$,
(b) $\mathbf{u} \cdot \mathbf{u} = 0$ iff $\mathbf{u} = \mathbf{0}$.
Solution:

(a) (2 points) $\mathbf{u} \cdot \mathbf{u} = u_1^2 + \cdots + u_n^2 \geq 0 + \cdots + 0 = 0$
(b) (3 points) ($\Leftarrow$) If $\mathbf{u} = \mathbf{0}$, then clearly $\mathbf{u} \cdot \mathbf{u} = 0$. ($\Rightarrow$) Assume $\mathbf{u} \cdot \mathbf{u} = 0$. Then $u_1^2 + \cdots + u_n^2 = 0$. Since the sum of nonnegative numbers $u_1^2, \ldots, u_n^2$ is zero, every summand u_i^2 is 0. Thus all u_i are 0, and hence $\mathbf{u} = \mathbf{0}$.

□

(102) [1, Section 6.1] Let $\mathbf{u} \in \mathbb{R}^n$. Is

$$V = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{u} \cdot \mathbf{x} = 0\}$$

a subspace of $\mathbb{R}^n$? Which conditions for a subspace are fulfilled by V?
Solution:
(1) The zero vector is in V since $\mathbf{u} \cdot \mathbf{0} = 0$.
(2) V is closed under addition: Let $\mathbf{x}, \mathbf{y} \in V$. Then $\mathbf{u} \cdot \mathbf{x} = 0$ and $\mathbf{u} \cdot \mathbf{y} = 0$. Show that $\mathbf{x} + \mathbf{y} \in V$. We have $\mathbf{u} \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{u} \cdot \mathbf{x} + \mathbf{u} \cdot \mathbf{y} = 0 + 0 = 0$. Thus $\mathbf{x} + \mathbf{y} \in V$.
(3) V is closed under scalar multiplication: Let $c \in \mathbb{R}$. Show that $c\mathbf{x} \in V$. We have $\mathbf{u} \cdot (c\mathbf{x}) = c(\mathbf{u} \cdot \mathbf{x}) = c0 = 0$. Thus $c\mathbf{x} \in V$.

□

(103) [1, Section 5.3 (cf. Section 5.6, Problem 5)] Consider a population of owls feeding on a population of flying squirrels in a wood. In month k, let o_k denote the number

of owls and f_k the number of flying squirrels. Assume that the populations change every month as follows:

$$\begin{aligned} o_{k+1} &= 0.3o_k + 0.4f_k \\ f_{k+1} &= -0.4o_k + 1.3f_k \end{aligned}$$

That is, if there would be no squirrels to hunt, only 30% of the owls would survive to the next month; if there were no owls that hunted squirrels, then the squirrel population would grow by factor 1.3 every month.

Let $\mathbf{x}_k = \begin{bmatrix} o_k \\ f_k \end{bmatrix}$. Express the population change from $\mathbf{x}_k$ to $\mathbf{x}_{k+1}$ using a matrix A. Diagonalize A.

Solution:

$$\mathbf{x}_{k+1} = \underbrace{\begin{bmatrix} 0.3 & 0.4 \\ -0.4 & 1.3 \end{bmatrix}}_{A} \mathbf{x}_k$$

We diagonalize A. The characteristic equation is

$$0 = \det(A - \lambda I) = (0.3 - \lambda)(1.3 - \lambda) + 0.4^2 = \lambda^2 - 1.6\lambda + 0.55,$$

the eigenvalues are $\lambda = \frac{1}{2}(1.6 \pm \sqrt{1.6^2 - 4 \cdot 0.55}) = 0.8 \pm 0.3 \in \{0.5, 1.1\}$. We compute a basis for each eigenspace.

$$\lambda = 0.5: \quad \mathrm{Nul}(A - 0.5I) = \mathrm{Nul}\begin{bmatrix} -0.2 & 0.4 \\ -0.4 & 0.8 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} = \mathrm{Span}\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\}.$$

$$\lambda = 1.1: \quad \mathrm{Nul}(A - 1.1I) = \mathrm{Nul}\begin{bmatrix} -0.8 & 0.4 \\ -0.4 & 0.2 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} = \mathrm{Span}\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\}.$$

We write the eigenvectors in a matrix P and compute P^{-1}:

$$P = \begin{bmatrix} 2 & 1/2 \\ 1 & 1 \end{bmatrix}, \qquad P^{-1} = \frac{1}{3/2}\begin{bmatrix} 1 & -1/2 \\ -1 & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -2 & 4 \end{bmatrix}.$$

We obtain a diagonalization

$$A = \underbrace{\begin{bmatrix} 2 & 1/2 \\ 1 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0.5 & 0 \\ 0 & 1.1 \end{bmatrix}}_{D} \underbrace{\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -2 & 4 \end{bmatrix}}_{P^{-1}}.$$

□

(104) Continue the previous problem: Let the starting population be $\mathbf{x}_1 = \begin{bmatrix} o_1 \\ f_1 \end{bmatrix} = \begin{bmatrix} 20 \\ 100 \end{bmatrix}$.

(a) Give an explicit formula for the populations in month $k+1$.
(b) Are the populations growing or decreasing over time? By which factor?
(c) What is ratio of owls to squirrels after 12 months? After 24 months? Can you explain why?

Solution:

(a) (2 points)

$$\mathbf{x}_{k+1} = A^k \mathbf{x}_1 = PD^kP^{-1}\mathbf{x}_1 = \begin{bmatrix} 2 & 1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0.5^k & 0 \\ 0 & 1.1^k \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 20 \\ 100 \end{bmatrix} = \begin{bmatrix} 60 \cdot 1.1^k - 40 \cdot 0.5^k \\ 120 \cdot 1.1^k - 20 \cdot 0.5^k \end{bmatrix}$$

(b) (2 points) Both populations are growing. For large k, the term 0.5^k can be neglected (e.g. for $k \geq 12$ we have $1.1^k \geq 3.138$ and $0.5^k \leq 0.00025$). We can approximate the populations by

$$\mathbf{x}_{k+1} \approx \begin{bmatrix} 60 \cdot 1.1^k \\ 120 \cdot 1.1^k \end{bmatrix} = 1.1^k \begin{bmatrix} 60 \\ 120 \end{bmatrix} \quad \text{for large } k.$$

After a large number of months, both populations grow by a factor of 1.1 every month.

(c) (1 point) The populations are $x_{13} = \begin{bmatrix} 188.3 \\ 376.6 \end{bmatrix}$ after 12 months and $x_{25} = \begin{bmatrix} 591.0 \\ 1182.0 \end{bmatrix}$ after 24 months. After a large number of months, the ratio of owls to squirrels is always about 1 : 2 by the approximation formula for $\mathbf{x}_{k+1}$.

□

(105) [1, cf. Section 6.1, Problems 19, 20] Are the following true or false? Why? All vectors are in $\mathbb{R}^n$.

(a) $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$.
(b) For any scalar c, $\mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$.
(c) For a square matrix A, vectors in $\operatorname{Col} A$ are orthogonal to vectors in $\operatorname{Nul} A$.
(d) If $\mathbf{x}$ is orthogonal to every vector in $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$, then $\mathbf{x}$ is also orthogonal to every vector in $\operatorname{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.
(e) If $||\mathbf{u}||^2 + ||\mathbf{v}||^2 = ||\mathbf{u} + \mathbf{v}||^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

Solution:

(a) True since $\mathbf{v} \cdot \mathbf{v} = v_1^2 + \cdots + v_n^2 = ||\mathbf{v}||^2$.
(b) True since $\mathbf{u} \cdot (c\mathbf{v}) = u_1cv_1 + \cdots + u_ncv_n = cu_1v_1 + \cdots + cu_nv_n = c(\mathbf{u} \cdot \mathbf{v})$.
(c) False. The correct statement is: For a square matrix A, vectors in $\operatorname{Row} A$ are orthogonal to vectors in $\operatorname{Nul} A$.
(d) True by the yellow box after Theorem 6.2.
(e) True by the Pythagorean Theorem (6.2).

□

(106) [1, Section 6.1] Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$.

(a) Compute the distance between $\mathbf{u}_1$ and $\mathbf{u}_2$ as well as between $\mathbf{u}_2$ and $\mathbf{u}_3$.
(b) Compute the angle (in degrees) between $\mathbf{u}_1$ and $\mathbf{u}_2$ as well as between $\mathbf{u}_2$ and $\mathbf{u}_3$.

Solution:

(a) $\mathrm{dist}(\mathbf{u}_1, \mathbf{u}_2) = ||\mathbf{u}_1 - \mathbf{u}_2|| = || \begin{bmatrix} -4 \\ 3 \\ 4 \end{bmatrix} || = \sqrt{4^2 + 3^2 + 4^2} = \sqrt{41}.$

$\mathrm{dist}(\mathbf{u}_2, \mathbf{u}_3) = ||\mathbf{u}_2 - \mathbf{u}_3|| = || \begin{bmatrix} 3 \\ -1 \\ -4 \end{bmatrix} || = \sqrt{3^2 + 1^2 + 4^2} = \sqrt{26}.$

(b) $\alpha(\mathbf{u}_1, \mathbf{u}_2) = \arccos \frac{\mathbf{u}_1 \cdot \mathbf{u}_2}{||\mathbf{u}_1|| \cdot ||\mathbf{u}_2||} = \arccos 0 = 90°$

$\alpha(\mathbf{u}_2, \mathbf{u}_3) = \arccos \frac{\mathbf{u}_2 \cdot \mathbf{u}_3}{||\mathbf{u}_2|| \cdot ||\mathbf{u}_3||} = \arccos \frac{7}{\sqrt{27}\sqrt{13}} \approx 68.06°$

□

(107) [1, Section 6.2] Let $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 6 \\ -1 \\ 1 \end{bmatrix}.$

(a) Verify that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set.

(b) Write every unit vector $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^3$ as linear combination $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$.

Solution:

(a) (1 point) $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, $\mathbf{u}_1 \cdot \mathbf{u}_3 = 0$, $\mathbf{u}_2 \cdot \mathbf{u}_3 = 0$.

(b) (4 points)

$$\mathbf{e}_1 = \frac{\mathbf{e}_1 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{e}_1 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{e}_1 \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3 = \frac{0}{2}\mathbf{u}_1 + \frac{1}{19}\mathbf{u}_2 + \frac{6}{38}\mathbf{u}_3,$$
$$\mathbf{e}_2 = \frac{\mathbf{e}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{e}_2 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{e}_2 \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3 = \frac{1}{2}\mathbf{u}_1 + \frac{3}{19}\mathbf{u}_2 + \frac{-1}{38}\mathbf{u}_3,$$
$$\mathbf{e}_3 = \frac{\mathbf{e}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{e}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{e}_3 \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3 = \frac{1}{2}\mathbf{u}_1 + \frac{-3}{19}\mathbf{u}_2 + \frac{1}{38}\mathbf{u}_3.$$

□

(108) [1, Section 6.1] Let $V = \mathrm{Span}\{\begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix}\}$ be a subspace of $\mathbb{R}^3$. Compute the orthogonal complement of V.

Solution:

Since V is the row space of the matrix $A = \begin{bmatrix} 1 & 6 & -1 \\ -3 & 1 & 3 \end{bmatrix}$, the orthogonal complement $V^\perp$ is given by

$$V^\perp = (\mathrm{Row}\, A)^\perp = \mathrm{Nul}\, A = \mathrm{Nul} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \mathrm{Span}\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\}.$$

□

References

[1] David C. Lay. Linear Algebra and Its Applications. Addison-Wesley, 4th edition, 2012.