Math 3130 - Assignment 11

Due April 8, 2016

(91) [1, Section 5.1] Are the following eigenvalues for the respective matrices? If so, give a basis for the corresponding eigenspace.

(a) $A = \begin{bmatrix} -4 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & 3 & -2 \end{bmatrix}, \lambda = -5$ (b) $B = \begin{bmatrix} 3 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \mu = 2$

Solution:

(a) $\mathrm{Nul}(A - (-5)I)$ has basis $\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right)$. So -5 is an eigenvalue of A.

(b) $\mathrm{Nul}(B - 2I) = 0$. So 2 is not an eigenvalue of B.

□

(92) [1, Section 5.1] Give all eigenvalues and bases for eigenspaces. Do you need the characteristic polynomials?

(a) $A = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 3 \end{bmatrix}$

Solution:

Since A, B are triangular matrices, their eigenvalues are just their diagonal elements.

(a) A has eigenvalue -3 with multiplicity 2: $\mathrm{Nul}(A - (-3)I)$ has basis $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$

(b) B has eigenvalues $2, 0, 3$:

$\mathrm{Nul}(A - 2I)$ has basis $\left(\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right)$.

$\mathrm{Nul}(A - 0I)$ has basis $\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$.

$\mathrm{Nul}(A - 3I)$ has basis $\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$.

□

(93) [1, Section 5.2] Give the characteristic polynomial, all eigenvalues and bases for eigenspaces for $C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$.

Solution:

Characteristic polynomial:

$$\begin{aligned} \det(C - \lambda I) &= (1 - \lambda)(1 - \lambda) - 2 \cdot 3 \\ &= \lambda^2 - 2\lambda - 5 \end{aligned}$$

Eigenvalues are the roots of the characteristic polynomial. Use the quadratic formula

$$\begin{aligned}\lambda_{1,2} &= 1 \pm \sqrt{1+5}\\ &= 1 \pm \sqrt{6}\end{aligned}$$

Eigenvector for $\lambda = 1 + \sqrt{6}$:

$$C - \lambda I = \begin{bmatrix} -\sqrt{6} & 2 \\ 3 & -\sqrt{6} \end{bmatrix} \sim \begin{bmatrix} -\sqrt{6} & 2 \\ 0 & 0 \end{bmatrix}$$

where we multiplied row 1 by $\frac{3}{\sqrt{6}}$ and added to row 2.

So the eigenspace for $\lambda = 1 + \sqrt{6}$ has basis $(\begin{bmatrix} 2 \\ \sqrt{6} \end{bmatrix})$.

Eigenvector for $\lambda = 1 - \sqrt{6}$:

$$C - \lambda I = \begin{bmatrix} \sqrt{6} & 2 \\ 3 & \sqrt{6} \end{bmatrix} \sim \begin{bmatrix} \sqrt{6} & 2 \\ 0 & 0 \end{bmatrix}$$

where we multiplied row 1 by $\frac{3}{\sqrt{6}}$ and subtracted from row 2.

So the eigenspace for $\lambda = 1 - \sqrt{6}$ has basis $(\begin{bmatrix} -2 \\ \sqrt{6} \end{bmatrix})$.

□

(94) [1, Section 5.2] Compute eigenvalues and eigenvectors for $D = \begin{bmatrix} -1 & 4 & 1 \\ 6 & 9 & 2 \\ 0 & 0 & -3 \end{bmatrix}$.

Solution:
Characteristic polynomial:

$$\begin{aligned}\det(D - \lambda I) = (-3 - \lambda) \cdot \det \begin{bmatrix} -1-\lambda & 4 \\ 6 & 9-\lambda \end{bmatrix}\\ (-3-\lambda)[(-1-\lambda)(9-\lambda) - 24]\\ (-3-\lambda)[\lambda^2 - 8\lambda - 33]\end{aligned}$$

Eigenvalues are $\lambda_1 = -3$ and the roots of $\lambda^2 - 8\lambda - 33$. The quadratic formula yields

$$\lambda_{2,3} = 4 \pm \sqrt{4^2 + 33}$$

So $\lambda_2 = -3$ and $\lambda_3 = 11$.

The eigenspace for $\lambda = -3$ has basis $(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix})$.

The eigenspace for $\lambda = 11$ has basis $(\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix})$. □

(95) [1, Section 5.1-5.2] Are the following true or false? Why?

(a) The eigenvalues of a matrix are its diagonal entries.
False but true for triangular matrices.

(b) A matrix is invertible iff 0 is not an eigenvalue.
True because 0 is an eigenvalue iff $\det(A - 0I) = 0$ iff A is not invertible.

(c) Any two matrices with the same characteristic polynomial have the same eigenvalues.
True because eigenvalues are the roots of the characteristic polynomial.
(d) Any two matrices with the same characteristic polynomial have the same eigenvectors.
False by an example in class.
(e) If two matrices have the same eigenvalues, then they are equal.
False by an example in class.

(96) [1, Section 5.3] Are the matrices A, B, C, D in 92, 93, 94 diagonalizable? How?
Solution:
A is not diagonalizable because its eigenvalue -3 has multiplicity 2 but the corresponding eigenspace only dimension 1.
B is diagonalizable because it has 3 distinct eigenvalues, so

$$B = P \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} P^{-1} \text{ for } P = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

C is diagonalizable because it has 2 distinct eigenvalues, so

$$C = P \begin{bmatrix} 1+\sqrt{6} & 0 \\ 0 & 1-\sqrt{6} \end{bmatrix} P^{-1} \text{ for } P = \begin{bmatrix} 2 & -2 \\ \sqrt{6} & \sqrt{6} \end{bmatrix}$$

D is not diagonalizable because its eigenvalue -3 has multiplicity 2 but the corresponding eigenspace only dimension 1. □

(97) [1, Section 5.3] Let A be an $n \times n$-matrix. Are the following true or false? Why?
(a) If A has n eigenvectors, then A is diagonalizable.
False. You need n linearly independent eigenvectors.
(b) If a 4×4-matrix A has two eigenvalues with eigenspaces of dimension 3 and 1, respectively, then A is diagonalizable.
True.
(c) If A is invertible, then A is diagonalizable.
False. See for example A in problem 92.
(d) A is diagonalizable iff A has n eigenvalues (counting multiplicities).
False. See for example A in problem 92.
(e) If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.
True. A basis of $\mathbb{R}^n$ of eigenvectors consists of n linearly independent eigenvectors.

(98) [1, Section 5.3] Let A be the standard matrix for the reflection of $\mathbb{R}^2$ on some line g throught the origin. What are the eigenvalues and eigenvectors of A? Can A be diagonalized? Hint: Consider what a reflection does to specific vectors.
Solution:
Let v_1 be a nonzero vector on the line g. Its reflection is again v_1. So

$$Av_1 = v_1$$

and v_1 is an eigenvector of A for eigenvalue 1.

Let v_2 be a nonzero vector perpendicular to the line g. Its reflection is $-v_2$. So

$$Av_2 = -v_2$$

and v_2 is an eigenvector of A for eigenvalue -1.

Hence

$$A = P \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} P^{-1}$$

for P with columns v_1 and v_2. □

(99) [1, Section 5.2] Let A be an $n \times n$-matrix with n eigenvalues $\lambda_1, \dots, \lambda_n$ (repeated according to their multiplicities). Show that

$$\det A = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$$

Hint: Consider the characteristic polynomial $\det(A - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$.

Solution:

Since $\lambda_1, \dots, \lambda_n$ are the roots of the characteristic polynomial, the characteristic polynomial can be factored as

$$\det(A - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda).$$

Note that the signs are correct because on both sides of the equation the coefficient of λ^n is $(-1)^n$.

By plugging in $\lambda = 0$ we get

$$\det A = \lambda_1 \cdot \lambda_2 \cdots \lambda_n.$$

□

References

[1] David C. Lay. Linear Algebra and Its Applications. Addison-Wesley, 4th edition, 2012.