Math 3130 - Assignment 10

Due April 1, 2016
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(81) [1, Section 4.3] Let A be an $n \times n$ matrix. Is

$$H = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = 2\mathbf{x}\}$$

a subspace of $\mathbb{R}^n$? Which conditions for a subspace are fulfilled by H?
Solution:
Yes. We show the subspace conditions. (1) Since $A\mathbf{0} = 2\mathbf{0}$, the zero vector is in H. (2) Let $\mathbf{u}, \mathbf{v} \in H$. Then $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u}+A\mathbf{v} = 2\mathbf{u}+2\mathbf{v} = 2(\mathbf{u}+\mathbf{v})$. Thus $\mathbf{u}+\mathbf{v} \in H$. (3) Let $r \in \mathbb{R}$. Then $A(r\mathbf{u}) = rA\mathbf{u} = r2\mathbf{u} = 2(r\mathbf{u})$. Thus $r\mathbf{u} \in H$. □

(82) [1, Section 4.3] Let $\mathbf{u}, \mathbf{v}$ be linearly independent vectors in a vector space V.
(a) Find all $x_1, x_2 \in \mathbb{R}$ such that

$$x_1(\mathbf{u}+\mathbf{v}) + x_2(\mathbf{u}-\mathbf{v}) = 0.$$

(b) Are the vectors $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ linearly independent?
Solution:

(a) (4 points) We have

$$(x_1+x_2)\mathbf{u} + (x_1-x_2)\mathbf{v} = \mathbf{0}.$$

Since $\mathbf{u}, \mathbf{v}$ are linearly independent, the coefficients x_1+x_2 and x_1-x_2 have to be zero:

$$x_1 + x_2 = 0,$$
$$x_1 - x_2 = 0.$$

The solution of this system is $x_1 = x_2 = 0$.
(b) (1 points) The equation from (a) has only the trivial solution. Thus $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$ are linearly independent.

□

(83) [1, Section 4.4] Let $B = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) = (1+t, 1+t^2, t+t^2)$ be a basis of $\mathbb{P}_2$, and let $\mathbf{u} = 1+t^2$ and $\mathbf{v} = 2t$.
(a) Write both $\mathbf{u}$ and $\mathbf{v}$ as linear combination of $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$.
(b) Give the B-coordinates $[\mathbf{u}]_B$ and $[\mathbf{v}]_B$.
Solution:

(a) (4 points) Since $\mathbf{u}$ is the second basis vector, we have $\mathbf{u} = \mathbf{b}_2$.
We solve $x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3 = \mathbf{v}$, i.e.,

$$x_1(1+t) + x_2(1+t^2) + x_3(t+t^2) = 2t.$$

We expand and collect terms with equal powers of t:

$$\underbrace{(x_1+x_2)}_{0} + \underbrace{(x_1+x_3)}_{2}\, t + \underbrace{(x_2+x_3)}_{0}\, t^2 = 0 + 2t + 0t^2.$$

This yields a linear system in x_1, x_2, x_3. We reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

The solution is $(x_1, x_2, x_3) = (1, -1, 1)$. Thus $\mathbf{v} = \mathbf{b}_1 - \mathbf{b}_2 + \mathbf{b}_3$.

(b) (1 point)

$$[\mathbf{u}]_B = [\mathbf{b}_2]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [\mathbf{v}]_B = [\mathbf{b}_1 - \mathbf{b}_2 + \mathbf{b}_3]_B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

□

(84) For which $\lambda \in \mathbb{R}$ is

$$\lambda(\lambda^2 - 2)(\lambda^2 + 1)(\lambda^2 - 3\lambda + 2) = 0? \qquad (1)$$

Solution:

The product is zero iff at least one of the factors is zero. Case 1, $\lambda = 0$.

Case 2, $\lambda^2 - 2 = 0$. This means $\lambda^2 = 2$, and thus λ is $\sqrt{2}$ or $-\sqrt{2}$.

Case 3, $\lambda^2 + 1 = 0$. This case cannot apply since $\lambda^2 + 1$ is always ≥ 1.

Case 4, $\lambda^2 - 3\lambda + 2 = 0$. The formula for quadratic equations yields

$$\lambda = \frac{3 \pm \sqrt{3^2 - 4 \cdot 2}}{2} = \frac{3}{2} \pm \frac{1}{2} \in \{1, 2\}.$$

Answer: Formula (1) holds iff $\lambda \in \{0, 1, 2, \sqrt{2}, -\sqrt{2}\}$. □

(85) [1, Section 3.2] For which $\mu \in \mathbb{R}$ has the matrix

$$B = \begin{bmatrix} 6 - \mu & 2 \\ -6 & -1 - \mu \end{bmatrix}$$

a determinant $\det B = 0$?

Solution:

$$\det B = (6 - \mu)(-1 - \mu) - (-12) = \mu^2 + \mu - 6\mu - 6 + 12 = \mu^2 - 5\mu + 6.$$

Now $\det B = 0$ yields a quadratic equation $\mu^2 - 5\mu + 6 = 0$ whose solution is

$$\mu = \frac{5 \pm \sqrt{5^2 - 4 \cdot 6}}{2} = \frac{5}{2} \pm \frac{1}{2} \in \{2, 3\}.$$

Thus $\det B = 0$ iff $\mu \in \{2, 3\}$. □

(86) [1, Section 4.2] Let

$$A = \begin{bmatrix} 6 & 2 \\ -6 & -1 \end{bmatrix}.$$

(a) Compute the matrices $A - 2I$, $A - 3I$, and $A - I$.

(b) Find all $\mathbf{x} \in \mathbb{R}^2$ such that $A\mathbf{x} = 2\mathbf{x}$. Give the parametric vector form for the solution set.
Hint: $A\mathbf{x} = 2\mathbf{x}$ iff $A\mathbf{x} = 2I\mathbf{x}$ iff $(A - 2I)\mathbf{x} = \mathbf{0}$.

(c) Find all $\mathbf{x} \in \mathbb{R}^2$ such that $A\mathbf{x} = 3\mathbf{x}$. Give the parametric vector form.

(d) Find all $\mathbf{x} \in \mathbb{R}^2$ such that $A\mathbf{x} = \mathbf{x}$. Give the parametric vector form.

Solution:

(a)

$$A - 2I = \begin{bmatrix} 4 & 2 \\ -6 & -3 \end{bmatrix}, \quad A - 3I = \begin{bmatrix} 3 & 2 \\ -6 & -4 \end{bmatrix}, \quad A - I = \begin{bmatrix} 5 & 2 \\ -6 & -2 \end{bmatrix}.$$

(b) We solve $(A - 2I)\mathbf{x} = \mathbf{0}$ and obtain $\mathbf{x} = r \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ for $r \in \mathbb{R}$.

(c) We solve $(A - 3I)\mathbf{x} = \mathbf{0}$ and obtain $\mathbf{x} = r \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$ for $r \in \mathbb{R}$.

(d) We solve $(A - I)\mathbf{x} = \mathbf{0}$ and obtain $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

□

(87) [1, Section 3.2] For which $\lambda \in \mathbb{R}$ has the matrix

$$B = \begin{bmatrix} -2-\lambda & 0 & 2 \\ 6 & 2-\lambda & -3 \\ -6 & 0 & 5-\lambda \end{bmatrix}$$

a determinant $\det B = 0$?

Solution:

Cofactor expansion across the second column yields

$$\det B = (2-\lambda)\det \begin{bmatrix} -2-\lambda & 2 \\ -6 & 5-\lambda \end{bmatrix} = (2-\lambda)(\lambda^2 - 3\lambda + 2).$$

Now $\det B = 0$ iff $\lambda = 2$ or $\lambda^2 - 3\lambda + 2 = 0$. The quadratic equation yields

$$\lambda = \frac{3 \pm \sqrt{3^2 - 4 \cdot 2}}{2} = \frac{3}{2} \pm \frac{1}{2} \in \{1, 2\}.$$

Hence $\det B = 0$ iff $\lambda \in \{1, 2\}$. □

(88) [1, Section 4.2] Let

$$A = \begin{bmatrix} -2 & 0 & 2 \\ 6 & 2 & -3 \\ -6 & 0 & 5 \end{bmatrix}$$

(a) Compute the matrices $A - 2I$ and $A - I$.

(b) Find all $\mathbf{x} \in \mathbb{R}^3$ such that $A\mathbf{x} = 2\mathbf{x}$. Give the parametric vector form for the solution set.

(c) Find all $\mathbf{x} \in \mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{x}$. Give the parametric vector form.

Solution:

(a)

$$A - 2I = \begin{bmatrix} -4 & 0 & 2 \\ 6 & 0 & -3 \\ -6 & 0 & 3 \end{bmatrix}, \quad A - I = \begin{bmatrix} -3 & 0 & 2 \\ 6 & 1 & -3 \\ -6 & 0 & 4 \end{bmatrix}.$$

(b) We solve $(A - 2I)\mathbf{x} = \mathbf{0}$ and reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} -4 & 0 & 2 & 0 \\ 6 & 0 & -3 & 0 \\ -6 & 0 & 3 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We obtain

$$\mathbf{x} = r \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \quad \text{for } r, s \in \mathbb{R}.$$

(c) We solve $(A - I)\mathbf{x} = \mathbf{0}$ and reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} -3 & 0 & 2 & 0 \\ 6 & 1 & -3 & 0 \\ -6 & 0 & 4 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We obtain

$$\mathbf{x} = r \begin{bmatrix} \frac{2}{3} \\ -1 \\ 1 \end{bmatrix} \quad \text{for } r \in \mathbb{R}.$$

□

References

[1] David C. Lay. Linear Algebra and Its Applications. Addison-Wesley, 4th edition, 2012.