Math 3130 - Assignment 9

Due March 18, 2016
Markus Steindl

(73) [1, Section 4.2] Let $T\colon \mathbb{P}_3 \to \mathbb{R}, p \mapsto p(3)$, be the map that evaluates a polynomial p at $x = 3$.

(a) Show that T is linear.
(b) Determine the kernel and the range of T.
(c) Is T injective, surjective, bijective?

Solution:

(a) For linearity, let $p, q \in \mathbb{P}_3$. Their sum $p + q$ is the polynomial that maps t to $p(t) + q(t)$. So

$$T(p+q) = (p+q)(3) = p(3) + q(3) = T(p) + T(q).$$

Further let $c \in \mathbb{R}$. Then cp maps t to $cp(t)$. So

$$T(cp) = (cp)(3) = cp(3) = cT(p).$$

Hence T is linear.

(b) The kernel of T, $\ker T$, consists of all the polynomials that evaluate to 0 at 3, that is,

$$\ker T = \{(t-3)q : q \in \mathbb{P}_2\}.$$

The range of T, $T(\mathbb{P}_3)$, is $\mathbb{R}$. For every $b \in \mathbb{R}$, there exists a polynomial $p \in \mathbb{P}_3$ that is mapped to b. Choose for example the constant polynomial $p(t) = b$.

(c) Since the kernel of T is non-trivial, T is not injective.
Since the range of T is equal to its codomain, T is surjective.
T is not bijective since it is not injective.

$\square$

(74) [1, Section 4.4]

(a) Let $B = \left(\begin{bmatrix}1\\1\\3\end{bmatrix}, \begin{bmatrix}2\\-2\\1\end{bmatrix}\right)$ be a basis of a subspace H of $\mathbb{R}^3$. Compute the coordinates $[u]_B$ for $u = \begin{bmatrix}-5\\11\\5\end{bmatrix}$ in H.

Solution:
Solve the linear system

$$c_1 \begin{bmatrix}1\\1\\3\end{bmatrix} + c_2 \begin{bmatrix}2\\-2\\1\end{bmatrix} = \begin{bmatrix}-5\\11\\5\end{bmatrix}$$

to obtain $c_1 = 3, c_2 = -4$. So $[u]_B = \begin{bmatrix}3\\-4\end{bmatrix}$. $\square$

(b) Let $C = (1+t, t+t^2, 1+t^2)$ be a basis for $\mathbb{P}_2$. Compute the coordinates $[p]_C$ for $p = 2+t^2$.

Solution:

Solve

$$c_1(1+t) + c_2(t+t^2) + c_3(1+t^2) = 2+t^2.$$

Comparing the coefficients on both sides of this equation yields

$$\begin{aligned} c_1 + c_3 &= 2 && \text{(constant part)} \\ c_1 + c_2 &= 0 && \text{(multiples of } t\text{)} \\ c_2 + c_3 &= 1 && \text{(multiples of } t^2\text{)} \end{aligned}$$

Solving that system of linear equations yields $c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}, c_3 = \frac{3}{2}$. So $[u]_B = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$. □

(75) [1, Section 4.6]

(a) If A is a 3×4-matrix, what is the largest possible rank of A? What is the smallest possible dimension of Nul A?

Solution:

The rank of a matrix is the number of its pivot elements, which is at most the number of its rows and at most the number of its columns. So $\text{rank}A \leq \max(3,4) = 3$. Since the largest possible rank is 3, the smallest number of free variables in $Ax = 0$ is 1. So the dimension of Nul A is 1 or larger. □

(b) If the nullspace of a 4×6-matrix B has dimension 3, what is the dimension of the row space of B?

Solution:

$\dim \text{Nul } A + \dim \text{Row } A =$ the number of columns of A

So $\dim \text{Row}\, A = 6 - \dim \text{Nul}\, A = 6 - 3 = 3$. □

(76) [1, Sections 4.3-4.6] True or false? Explain your answers:

(a) Any plane in $\mathbb{R}^3$ is isomorphic to $\mathbb{R}^2$.

Solution:

False. Only planes through the origin are subspaces of $\mathbb{R}^3$. They are isomorphic to $\mathbb{R}^2$ by the coordinate mapping. □

(b) A basis for V is a linear independent set that is as large as possible.

Solution:

True. If B is a basis and you add another vector v to B, the new set will be linearly independent because v is a linear combination of the vectors in B.

On the other hand, assume B is a linearly independent set such that whenever any other vector v is added to B, then the new set is linearly dependent. Then any other vector must be a linear combination of the vectors in B. So B spans V and B is a basis. □

(c) If $v_1, \ldots, v_k$ are linearly independent in V, then $k \leq \dim V$.

Solution:
True. A linear independent set cannot have more elements than a basis of V. ☐

(d) If B is an echelon form of A, then the pivot columns of B are a basis for Col A.
Solution:
False. The pivot columns of A are a basis for Col A. ☐

(e) The row space of A^T is equal to the column space of A.
Solution:
True. Transposing a matrix A changes the columns of A to the rows of A^T. ☐

(77) [1, Section 3.1] Compute the determinant of the matrices by cofactor expansion. Pick a row or column that yields the least amount of computation:

$$A = \begin{bmatrix} 0 & 1 & -3 \\ 5 & 4 & -4 \\ 0 & -3 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 3 & 1 & 5 & 1 \\ 2 & 0 & 0 & 0 \\ 7 & 1 & -2 & 5 \end{bmatrix}.$$

Solution:
Expand $\det A$ down the first column:

$$\det A = 0 \cdot \det A_{11} - 5 \cdot \det A_{21} + 0 \cdot \det A_{31} = -5 \cdot \det \begin{bmatrix} 1 & -3 \\ -3 & -4 \end{bmatrix} = -5(1(-4) - (-3)(-3)) = 65$$

Expand $\det B$ across 3rd row:

$$\det B = 2 \cdot \det B_{13} = 2 \cdot \det \begin{bmatrix} 0 & -3 & 0 \\ 1 & 5 & 1 \\ 1 & -2 & 5 \end{bmatrix}$$

Expand across 1st row:

$$\det B_{13} = -1(-3) \det \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} = 3 \cdot (1 \cdot 5 - 1 \cdot 1) = 12$$

So $\det B = 2 \cdot 12 = 24$. ☐

(78) [1, Section 3.1] **Rule of Sarrus for the determinant of 3×3-matrices.** Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Prove that

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Hint: Expand $\det A$ across the first row.

Solution:

$$\begin{aligned}\det A &= a_{11}\cdot\det A_{11} - a_{12}\cdot\det A_{12} + a_{13}\cdot\det A_{13}\\ &= a_{11}\cdot\det\begin{bmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{bmatrix} - a_{12}\cdot\det\begin{bmatrix}a_{21} & a_{23}\\ a_{31} & a_{33}\end{bmatrix} + a_{13}\cdot\det\begin{bmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{bmatrix}\\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})\\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}\end{aligned}$$

□

(79) [1, Section 3.1] Give two 3×3-matrices with determinat 5. (Hint: triangular matrices.)
Solution:
Any triangular or diagonal matrix whose diagonal elements multiply to 5 will do, e.g.,

$$A = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 5\end{bmatrix}$$

□

(80) [1, Section 3.2] Compute the determinants by row reduction to echelon form:

$$A = \begin{bmatrix}3 & 3 & -3\\ 3 & 4 & -4\\ 2 & -3 & -5\end{bmatrix} \qquad B = \begin{bmatrix}1 & 3 & 2 & -4\\ 0 & 1 & 2 & -5\\ 2 & 7 & 6 & -3\\ -3 & -10 & -7 & 2\end{bmatrix}$$

Solution:

$$\begin{aligned}\det A &= 3\cdot\det\begin{bmatrix}1 & 1 & -1\\ 3 & 4 & -4\\ 2 & -3 & -5\end{bmatrix} && \text{factoring 3 from the first row}\\ &= 3\cdot\det\begin{bmatrix}1 & 1 & -1\\ 0 & 1 & -1\\ 0 & -5 & -3\end{bmatrix} && \text{subtracting multiples of the first row from the others}\\ &= 3\cdot\det\begin{bmatrix}1 & 1 & -1\\ 0 & 1 & -1\\ 0 & 0 & -8\end{bmatrix} && \text{adding 5 times the second row to the third}\\ &= 3\cdot 1\cdot 1\cdot(-8) = -24.\end{aligned}$$

$$\det B = \det \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & -10 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & -15 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -15 \\ 0 & 0 & 0 & 10 \end{bmatrix} \quad \text{flipped row 3 and 4}$$

$$= -1 \cdot 1 \cdot 1 \cdot 10 = -10.$$

□

(81) [1, Section 3.2] Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) How does switching the rows effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} c & d \\ a & b \end{bmatrix}$.

Solution:
Interchanging 2 rows changes the sign of the determinant:

$$\det \begin{bmatrix} c & d \\ a & b \end{bmatrix} = cb - ad = -\det A$$

□

(b) How does adding a multiple of one row to the other row effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} a & b \\ c + ra & d + rb \end{bmatrix}$.

Solution:
Adding a multiple of the first row to another does not change the determinant:

$$\det \begin{bmatrix} a & b \\ c + ra & d + rb \end{bmatrix} = a(d + rb) - b(c + ra) = ad - bc = \det A$$

□

References

[1] David C. Lay. Linear Algebra and Its Applications. Addison-Wesley, 4th edition, 2012.