Math 3130 - Assignment 8

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(64) [1, Section 4.2] Let U, V, W be vector spaces, and let $f\colon U \to V$ and $g\colon V \to W$ be linear mappings.

(a) Show that the composition mapping $h\colon U \to W$, $\mathbf{u} \mapsto g(f(\mathbf{u}))$ is linear.
Solution:
Let $\mathbf{u}, \mathbf{v} \in U$ and $r \in \mathbb{R}$. Then

$$\begin{aligned} h(\mathbf{u}+\mathbf{v}) &= g(f(\mathbf{u}+\mathbf{v})) \\ &= g(f(\mathbf{u}) + f(\mathbf{v})) \quad \text{by linearity of } f, \\ &= g(f(\mathbf{u})) + g(f(\mathbf{v})) \quad \text{by linearity of } g, \\ &= h(\mathbf{u}) + h(\mathbf{v}). \end{aligned}$$

Also,

$$\begin{aligned} h(r\mathbf{u}) &= g(f(r\mathbf{u})) \\ &= g(rf(\mathbf{u})) \quad \text{by linearity of } f, \\ &= rg(f(\mathbf{u})) \quad \text{by linearity of } g, \\ &= rh(\mathbf{u}). \end{aligned}$$

Thus h is linear. □

(b) Does it make sense to ask whether $k\colon V \to V$, $\mathbf{u} \mapsto f(g(\mathbf{v}))$ is linear?
Solution:
No. The vector $g(\mathbf{v})$ is in W, but we do not know if $g(\mathbf{v})$ is in U. Thus $g(\mathbf{v})$ cannot be an input of f. □

(65) [1, Section 4.2] Let U, V be vector spaces and $T\colon U \to V$ be a linear mapping. Show that $T(\mathbf{0}) = \mathbf{0}$.
Hint: Write down $T(\mathbf{0}+\mathbf{0})$ in two different ways.
Solution:
We have $T(\mathbf{0}) = T(\mathbf{0}+\mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$. So

$$T(\mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0}).$$

By subtracting $T(\mathbf{0})$ from each side we obtain $\mathbf{0} = T(\mathbf{0})$. □

(66) [1, Section 4.2] Let U, V be vector spaces and $T\colon U \to V$ be a linear mapping. Show that the range $\operatorname{Rg} T$ is a subspace of V.
Solution:
We show the subspace conditions. (1) By problem (65) $T(\mathbf{0}) = \mathbf{0}$. Thus the zero vector is in the range.
(2) and (3): Let $\mathbf{v}_1, \mathbf{v}_2 \in \operatorname{Rg} T$ and $r \in \mathbb{R}$. We have to show that $\mathbf{v}_1+\mathbf{v}_2 \in \operatorname{Rg} T$ and $r\mathbf{v}_1 \in \operatorname{Rg} T$. Since $\mathbf{v}_1, \mathbf{v}_2$ are in the range, there are $\mathbf{u}_1, \mathbf{u}_2 \in U$ such that $T(\mathbf{u}_1) = \mathbf{v}_1$

and $T(\mathbf{u}_2) = \mathbf{v}_2$. Now

$$\mathbf{v}_1 + \mathbf{v}_2 = T(\mathbf{u}_1) + T(\mathbf{u}_2) = T(\mathbf{u}_1 + \mathbf{u}_2).$$

This means $\mathbf{v}_1 + \mathbf{v}_2$ is the image of $\mathbf{u}_1 + \mathbf{u}_2$. Thus $\mathbf{v}_1 + \mathbf{v}_2 \in \operatorname{Rg} T$.

Also,

$$r\mathbf{v}_1 = rT(\mathbf{u}_1) = T(r\mathbf{u}_1).$$

Thus $r\mathbf{v}_1$ is the image of $r\mathbf{u}_1$, and hence $r\mathbf{v}_1 \in \operatorname{Rg} T$. □

(67) [1, Section 4.3] Let $V = \{f : \mathbb{R} \to \mathbb{R}\}$ be a vector space of functions. Is the set $\{\cos t, \sin t, \sin(t + \frac{\pi}{4})\}$ linearly independent?

Hint: Use the formula for $\sin(\alpha + \beta)$.

Solution:

No. Since $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$, we have

$$\sin(t + \frac{\pi}{4}) = \sin(t)\cos(\frac{\pi}{4}) + \cos(t)\sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}\sin(t) + \frac{1}{\sqrt{2}}\cos(t).$$

Thus $\sin(t + \frac{\pi}{4})$ is a linear combination of $\sin(t)$ and $\cos(t)$. The vectors are not linearly independent. □

(68) [1, Section 4.4] Let $B = (\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix})$ and $C = (\begin{bmatrix}2\\5\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix})$ be bases of $\mathbb{R}^2$.

(a) Find the standard matrix for $f : \mathbb{R}^2 \to \mathbb{R}^2$, $[\mathbf{u}]_B \mapsto \mathbf{u}$.

Solution:

Since $\mathbf{u} = P_B[\mathbf{u}]_B$, the standard matrix of f is $P_B = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$. □

(b) Find the standard matrix for $g : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{u} \mapsto [\mathbf{u}]_C$.

Solution:

Since $[\mathbf{u}]_C = P_C^{-1}\mathbf{u}$, the standard matrix of g is $P_C^{-1} = \begin{bmatrix}2 & 1\\5 & 3\end{bmatrix}^{-1} = \begin{bmatrix}3 & -1\\-5 & 2\end{bmatrix}$. □

(c) Find the standard matrix for $h : \mathbb{R}^2 \to \mathbb{R}^2$, $[\mathbf{u}]_B \mapsto [\mathbf{u}]_C$. Hint: $h(\mathbf{x}) = g(f(\mathbf{x}))$.

Solution:

Since

$$h(\mathbf{x}) = g(f(\mathbf{x})) = g(P_B\mathbf{x}) = P_C^{-1}P_B\mathbf{x},$$

the standard matrix of h is $P_C^{-1}P_B = \begin{bmatrix}3 & -1\\-5 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \begin{bmatrix}2 & 4\\-3 & -7\end{bmatrix}$ □

(69) [1, Sections 4.1–4.4] Let $B = (t, 2 + t, t^2)$ and $C = (1, t + t^2, t^2)$ be bases of $\mathbb{P}_2$. Let h be the linear mapping $h : \mathbb{R}^3 \to \mathbb{R}^3$, $[\mathbf{u}]_B \mapsto [\mathbf{u}]_C$.

(a) Compute $h(\mathbf{e}_1), h(\mathbf{e}_2), h(\mathbf{e}_3)$. Hint: If $[\mathbf{u}]_B = \mathbf{e}_1$, then $[\mathbf{u}]_C = h(\mathbf{e}_1) = ?$

(b) Give the standard matrix of h.

(c) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{P}_2$ such that $[\mathbf{v}_1]_B = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $[\mathbf{v}_2]_B = \begin{bmatrix}2\\-1\\0\end{bmatrix}$, and $[\mathbf{v}_3]_B = \begin{bmatrix}0\\1\\2\end{bmatrix}$. Find $[\mathbf{v}_1]_C$, $[\mathbf{v}_2]_C$, $[\mathbf{v}_3]_C$.

Solution:
If $[\mathbf{u}]_B = \mathbf{e}_1$, then $\mathbf{u} = 1t + 0(2+t) + 0t^2 = t$. In order to compute $[\mathbf{u}]_C$, we have to find $x_1, x_2, x_3 \in \mathbb{R}$ such that $x_1 1 + x_2(t+t^2) + x_3 t^2 = \mathbf{u} = t$. I.e.,

$$x_1 1 + x_2 t + (x_2 + x_3)t^2 = 0 \cdot 1 + 1 \cdot t + 0 \cdot t^2.$$

□

(a) (2 points) We compare the coefficients of the LHS and RHS and obtain

$$x_1 = 0,\ x_2 = 1,\ x_2 + x_3 = 0.$$

This yields

$$h(\mathbf{e}_1) = [\mathbf{u}]_C = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

By similar computations, we obtain the following:
If $[\mathbf{u}]_B = \mathbf{e}_2$, then $\mathbf{u} = 2 + t$ and $h(\mathbf{e}_2) = [\mathbf{u}]_C = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

If $[\mathbf{u}]_B = \mathbf{e}_3$, then $\mathbf{u} = t^2$ and $h(\mathbf{e}_3) = [\mathbf{u}]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(b) (1 point) The standard matrix of h is given by

$$A = [h(\mathbf{e}_1)\ h(\mathbf{e}_2)\ h(\mathbf{e}_3)] = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}.$$

(c) (2 points)

$$[\mathbf{v}_1]_C = h([\mathbf{v}_1]_B) = A[\mathbf{v}_1]_B = A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix},$$

$$[\mathbf{v}_2]_C = h([\mathbf{v}_2]_B) = A[\mathbf{v}_2]_B = A \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix},$$

$$[\mathbf{v}_3]_C = h([\mathbf{v}_3]_B) = A[\mathbf{v}_3]_B = A \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

(70) [1, Section 4.3] Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 1 \\ 2.5 \\ -5 \end{bmatrix}$.

(a) Find vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ such that $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{u}_1, \ldots, \mathbf{u}_k)$ is a basis for $\mathbb{R}^3$.
(b) Find vectors $\mathbf{v}_1, \ldots, \mathbf{v}_\ell$ such that $(\mathbf{b}_3, \mathbf{v}_1, \ldots, \mathbf{v}_\ell)$ is a basis for $\mathbb{R}^3$.

Prove that your choices for (a) and (b) form a basis.

Solution:
Both bases have 3 vectors. Thus $k = 1$ and $\ell = 2$.

(a) One possible choice is $\mathbf{u}_1 = \mathbf{e}_1$. We show that $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{e}_1)$ is a basis by reducing the following augmented matrix to echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

There is no free variable. Thus the columns are linearly independent. Also, the vectors span $\mathbb{R}^3$ since there is no zero row.

(b) One possible choice is $\mathbf{v}_1 = \mathbf{e}_1$ and $\mathbf{v}_2 = \mathbf{e}_2$. We show that $(\mathbf{b}_3, \mathbf{e}_1, \mathbf{e}_2)$ is a basis by reducing the following augmented matrix to echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2.5 & 0 & 1 & 0 \\ -5 & 0 & 0 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

There is no free variable. Thus the columns are linearly independent. Also, the vectors span $\mathbb{R}^3$ since there is no zero row.

□

(71) [1, Sections 4.3, 4.5] Let

$$A = \begin{bmatrix} -5 & 8 & 0 & -17 & -2 \\ 3 & -5 & 1 & 5 & 1 \\ 11 & -19 & 7 & 1 & 3 \\ 7 & -13 & 5 & -3 & 1 \end{bmatrix}.$$

Find bases and dimensions for $\operatorname{Nul} A$, $\operatorname{Col} A$, and $\operatorname{Row} A$, respectively.

Solution:

(1 point) We reduce A to reduced echelon form:

$$A \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 5 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(2 points) For $\operatorname{Nul} A$, we solve $A\mathbf{x} = \mathbf{0}$ and obtain

$$\operatorname{Nul} A = \{r \begin{bmatrix} -5 \\ -1 \\ 5 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mid r, s \in \mathbb{R}\}.$$

The two vectors form a basis for $\operatorname{Nul} A$.

(1 point) The first three columns of A contain a pivot. Thus they form a basis

$$B = (\begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 8 \\ -5 \\ -19 \\ -13 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix})$$

for $\operatorname{Col} A$.

(1 point) The nonzero rows in any echelon form of A form a basis. E.g.,

$$C = (\begin{bmatrix}1\\0\\0\\5\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\-5\\0\end{bmatrix})$$

is a basis for $\operatorname{Row} A$. □

(72) [1, Section 4.5] A 177×35 matrix A has 19 pivots. Find $\dim \operatorname{Nul} A$, $\dim \operatorname{Col} A$, $\dim \operatorname{Row} A$, and $\operatorname{rank} A$.

Solution:

The number of pivots, $\dim \operatorname{Row} A$, $\dim \operatorname{Col} A$, and the rank are equal. So

$$\dim \operatorname{Row} A = \dim \operatorname{Col} A = \operatorname{rank} A = 19.$$

By the rank theorem, $\dim \operatorname{Nul} A + \operatorname{rank} A = 35$. Thus

$$\dim \operatorname{Nul} A = 35 - 19 = 16.$$

Thus □

References

[1] David C. Lay. Linear Algebra and Its Applications. Addison-Wesley, 4th edition, 2012.