Math 3130 - Assignment 7

Due March 4, 2016
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(55) [1, Sections 4.3, 4.4] Let $B = (b_1, \dots, b_n)$ be a basis for a vector space V and consider the coordinate mapping $V \to \mathbb{R}^n,\ x \mapsto [x]_B$.
(a) Show that $[c \cdot x]_B = c[x]_B$ for all $x \in V, c \in \mathbb{R}$.
(b) Show that the coordinate mapping is onto $\mathbb{R}^n$.
Solution:

(a) Let $x \in V$ with $x = c_1 b_1 + \dots c_n b_n$ for $c_1, \dots, c_n \in \mathbb{R}$. That is, $[x]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

Let $c \in \mathbb{R}$ and consider

$$cx = c(c_1 b_1 + \dots c_n b_n) = cc_1 b_1 + \dots cc_n b_n.$$

Then the coordinates of cx are

$$[cx]_B = \begin{bmatrix} cc_1 \\ \vdots \\ cc_n \end{bmatrix} = c \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c[x]_B.$$

(b) To show the map is onto $\mathbb{R}^n$, let $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$. We have to find $x \in V$ such that $[x]_B = y$. Pick $x = y_1 b_1 + \dots y_n b_n$. This shows that the coordinate map is onto. □

(56) [1, Section 4.2] Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most 2, and let $D : \mathbb{P}_2 \to \mathbb{P}_2, f \mapsto f'$, be the linear map that computes the derivative of a polynomial.
(a) Determine kernel and range of D.
(b) Is D injective, surjective, bijective?
Solution:

(a) The kernel of D ($\ker D$ or $\operatorname{Nul} D$) is the set of all constant polynomials.
The range of D is $D(\mathbb{P}_2) = \mathbb{P}_1$. Note that differentiating a polynomial of degree 2 yields a polynomial of degree 1. So $D(\mathbb{P}_2) \subseteq P_1$. Further every polynomial g of degree 1 is the derivative of a polynomial f of degree 2. Just integrate g to get f. So $D(\mathbb{P}_2) \supseteq P_1$ and hence $D(\mathbb{P}_2) = P_1$.
(b) D is not injective because $1' = 0 = 0'$. Alternatively, all constant polynomials are mapped to the same function (the kernel is non-trivial).
D is not surjective (onto $\mathbb{P}_2$) because its range is $D(\mathbb{P}_2) = P_1$ and not $\mathbb{P}_2$.
D is not bijective since it is neither injective nor surjective. □

(57) [1, Section 4.3] Which of the following are bases of $\mathbb{R}^3$? Why or why not?

$$A = (\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}), B = (\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}, \begin{bmatrix}0\\-1\\4\end{bmatrix}), C = (\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix})$$

Solution:
A is not a basis because 2 vectors can at most span a plane but not all of $\mathbb{R}^3$.

To check whether B is a basis we have to see whether it spans $\mathbb{R}^3$. Row reduce

$$\begin{bmatrix}1&2&0\\2&3&-1\\0&4&4\end{bmatrix} \sim \begin{bmatrix}1&2&0\\0&-1&-1\\0&0&0\end{bmatrix}$$

Since we have a 0-row, the vectors in B do not span $\mathbb{R}^3$. Hence B is not a basis.

To check whether C is a basis we have to see whether it spans $\mathbb{R}^3$. Row reduce

$$\begin{bmatrix}1&2&0\\2&3&1\\0&4&4\end{bmatrix} \sim \begin{bmatrix}1&2&0\\0&-1&1\\0&0&8\end{bmatrix}$$

The echelon form has no 0-row. So C spans $\mathbb{R}^3$. Further we see from the echelon form that C is linearly independent. So C is a basis. □

(58) [1, Section 4.3] Give a basis for Nul A and a basis for Col A for

$$A = \begin{bmatrix}0&2&0&3\\1&-4&-1&0\\-2&6&2&-3\end{bmatrix}$$

Solution:
Nul A is the solution set of $Ax = 0$. So we row reduce A

$$A = \begin{bmatrix}0&2&0&3\\1&-4&-1&0\\-2&6&2&-3\end{bmatrix} \sim \begin{bmatrix}1&-4&-1&0\\0&2&0&3\\0&-2&0&-3\end{bmatrix} \sim \begin{bmatrix}1&-4&-1&0\\0&2&0&3\\0&0&0&0\end{bmatrix}$$

to get the solution $x_4 = t, x_3 = s$ (both free), $x_2 = -\frac{3}{2}t, x_1 = s - 6t$. So

$$x = s\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + t\begin{bmatrix}-6\\-\frac{3}{2}\\0\\1\end{bmatrix} \text{ and Nul}\,A \text{ has basis } (\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-6\\-\frac{3}{2}\\0\\1\end{bmatrix})$$

For a basis of the column space Col$A$ we pick the pivot columns of A, i.e., the first and second column. So Col$A$ has basis $(\begin{bmatrix}0\\1\\-2\end{bmatrix}, \begin{bmatrix}2\\-4\\6\end{bmatrix})$. □

(59) [1, Section 4.3] Give 2 different bases for

$$H = \text{Span}\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}3\\-1\\0\end{bmatrix}, \begin{bmatrix}-1\\3\\4\end{bmatrix}\}$$

Solution:
Row reduction yields

$$\begin{bmatrix} 1 & 3 & -1 \\ 1 & -1 & 3 \\ 2 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

So the first 2 columns of the original matrix are pivot columns and form a basis of H.

$$B_1 = \left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \right)$$

Recall that the order of basis vectors is important (coordinates!). So by flipping the vectors we get a different basis

$$B_2 = \left(\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right)$$

Also note that any 2 columns of the matrix are linearly independent. So any 2 distinct vectors in any order out of $\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} \}$ form a basis. □

(60) [1, Section 4.3] Show that $\cos t, \cos 2t$ are linearly independent in the vector space of real valued functions.
Solution:
Show that

$$x_1 \cos t + x_2 \cos 2t = 0 \text{ (the constant 0-function)}$$

has only the trivial solution $x_1 = x_2 = 0$. Since this is an equation for functions, we want to look at it at various values. For $t = 0$, $t = \pi/2$, respectively, we get the equations

$$\begin{aligned} x_1 \cdot 1 + x_2 \cdot 1 &= 0, \\ x_1 \cdot 0 + x_2 \cdot (-1) &= 0. \end{aligned}$$

This system only has the trivial solution. So $\cos t, \cos 2t$ are linearly independent. □

(61) [1, Section 4.3] Consider the vector space of functions $V = \text{Span}\{\cos t, 2\cos t, \cos 2t, 3\cos 2t\}$. Give a basis for V.
Solution:
$2\cos t$ is a linear combination of $\cos t$, and $3\cos 2t$ a linear combination of $\cos 2t$. By the Spanning Set Theorem we do not need $2\cos t, 3\cos 2t$ in the spanning set for V. Since the remaining vectors $\cos t, \cos 2t$ are linearly independent by (60), they form a basis of V. □

(62) [1, Section 4.4] Let $B = \left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right)$ be a basis of $\mathbb{R}^2$.

(a) Give the change of coordinates matrix P_B from B to the standard basis $E = (e_1, e_2)$.

(b) Find vectors $u, v \in \mathbb{R}^2$ with $[u]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [v]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

(c) Compute the coordinates relative to B of $w = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution:

(a) For P_B we put the vectors of B in the columns of a matrix,

$$P_B = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$$

(b) $u = P_B \cdot [u]_B = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, the second vector in B

$v = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

(c) To find $[w]_B$ we can solve $P_B \cdot [w]_B = w$ directly by row reduction. Alternatively, we can invert P_B and use the formula $[w]_B = P_B^{-1} \cdot w$.

$$P_B^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

So

$$[w]_B = \frac{1}{-2} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$[x]_B = \frac{1}{-2} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

□

(63) [1, Sections 4.1–4.4] Let $B = (1, t, t^2)$ and $C = (1, 1+t, 1+t+t^2)$ be bases of $\mathbb{P}_2$.

(a) Determine the polynomials p, q with $[p]_B = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$ and $[q]_C = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$.

(b) Compute $[r]_B$ and $[r]_C$ for $r = 3 + 2t + t^2$.

Solution:

(a) $p = 3 - 2t^2$,
$q = 3 \cdot 1 + 0 \cdot (1+t) - 2(1+t+t^2) = 1 - 2t - 2t^2$

(b) For the coordinates relative to B just take the coefficients of the polynomial: $[r]_B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

For the coordinates relative to C consider the equation

$$\begin{aligned} r &= x_1 \cdot 1 + x_2(1+t) + x_3(1+t+t^2) \\ &= (x_1 + x_2 + x_3) + (x_2 + x_3)t + x_3 t^2 \end{aligned}$$

Comparing the coefficients we obtain $x_3 = 1, x_2 = 1, x_1 = 1$. So $[r]_C = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. ☐

References

[1] David C. Lay. Linear Algebra and Its Applications. Addison-Wesley, 4th edition, 2012.