Math 3130 - Assignment 6

Due February 26, 2016
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(46) Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors in a vector space V. Show that $H := \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a subspace of V.
Solution:
We show the 3 conditions for being a subspace.
(1) The zero vector can be written as linear combination $\mathbf{0} = 0\mathbf{v}_1 + \ldots + 0\mathbf{v}_n$. Thus $\mathbf{0} \in H$.
(2) Let $\mathbf{u}$ and $\mathbf{w}$ be arbitary vectors in H. We can write these vectors as

$$\mathbf{u} = a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n \quad \text{for some } a_1, \ldots, a_n \in \mathbb{R},$$
$$\mathbf{w} = b_1\mathbf{v}_1 + \ldots + b_n\mathbf{v}_n \quad \text{for some } b_1, \ldots, b_n \in \mathbb{R}.$$

Now

$$\begin{aligned}\mathbf{u} + \mathbf{w} &= a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n + b_1\mathbf{v}_1 + \ldots + b_n\mathbf{v}_n \\ &= (a_1 + b_1)\mathbf{v}_1 + \ldots + (a_n + b_n)\mathbf{v}_n.\end{aligned}$$

Thus $\mathbf{u} + \mathbf{w}$ is spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and hence an element of H.
(3) Let $\mathbf{u} \in H$ as above, and let $r \in \mathbb{R}$. Then

$$\begin{aligned}r\mathbf{u} &= r(a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n) \\ &= ra_1\mathbf{v}_1 + \ldots + ra_n\mathbf{v}_n.\end{aligned}$$

Thus $r\mathbf{u}$ is spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and hence an element of H.

□

(47) Let A be an $m \times n$ matrix. Prove that the Nullspace of A is a subspace of $\mathbb{R}^n$.
Solution:
We show the 3 conditions for being a subspace.
(1) The zero vector is clearly in $\text{Nul}(A)$ since $A\mathbf{0} = \mathbf{0}$.
(2) Let $\mathbf{u}$ and $\mathbf{w}$ be arbitary vectors in $\text{Nul}(A)$. Then $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{0}$. We show that $\mathbf{u} + \mathbf{w}$ is in $\text{Nul}(A)$.

$$A(\mathbf{u} + \mathbf{w}) = A\mathbf{u} + A\mathbf{w} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

So $\mathbf{u} + \mathbf{w}$ is in $\text{Nul}(A)$.
(3) Let $r \in \mathbb{R}$. Then

$$A(r\mathbf{u}) = r(A\mathbf{u}) = r\mathbf{0} = \mathbf{0}.$$

Hence $r\mathbf{u}$ is in $\text{Nul}(A)$.

□

(48) Let $M_{2\times 2}$ be the set of all 2×2 matrices. Let $+$ be the sum of matrices and $\cdot$ be the multiplication of a matrix by a scalar.

(a) Show that $M_{2\times 2}$ forms is a vector space.
Solution:
(1)–(3) The sum of two matrices is a matrix, and the addition is commutative and associative.
(4) The zero vector is the zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
(5) If $\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix, then $-\mathbf{u} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$.
(6)–(10) follow from several theorems on matrices. □

(b) Let H be the set of invertible 2×2 matrices. Show that H is not a subspace of $M_{2\times 2}$.
Solution:
(3 points) The zero vector $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is not an invertible matrix. Thus $\mathbf{0} \notin H$, and hence H is not a subspace. □

(49) Show that $V := \{\begin{bmatrix} a \\ b \end{bmatrix} \mid a, b \in \mathbb{R},\ a \geq 0\}$ is no subspace of $\mathbb{R}^2$.
Solution:
No. For example $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in V, but $(-1)\mathbf{u} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ is not in V. □

(50)

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 2 & 4 \\ 2 & -4 & 1 & 0 \\ -3 & 6 & 2 & 7 \end{bmatrix}$$

(a) Which of the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}$ are in Nul A?
Solution:
The vectors $\mathbf{u}$ and $\mathbf{v}$ are in Nul(A) since $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$. Vectors $\mathbf{w}$ and $\mathbf{x}$ cannot be in Nul(A) since they do not have 4 entries. □

(b) Which of the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}$ are in Col A?
Solution:
Since

$$\mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix},$$

$\mathbf{w}$ is spanned by the columns of A. Thus $\mathbf{w} \in$ Col(A). Also $\mathbf{0} \in$ Col(A). The vectors $\mathbf{u}$ and $\mathbf{v}$ cannot be in Col(A) since they have too many entries. □

(51) Let A be the matrix from (50).
(a) Solve $A\mathbf{x} = \mathbf{0}$ and give the solution in parametric vector form.

Solution:
We solve $A\mathbf{x} = \mathbf{0}$ and reduce the augmented matrix:

$$\left[\begin{array}{cccc|c} 0 & 0 & 2 & 4 & 0 \\ 2 & -4 & 1 & 0 & 0 \\ -3 & 6 & 2 & 7 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The variables x_2 and x_4 are free. We obtain

$$\begin{aligned} x_1 &= 2r + s \\ x_2 &= r \\ x_3 &= -2s \\ x_4 &= s. \end{aligned}$$

The solution in parametric vector form is

$$\mathbf{x} = r \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \qquad r, s \in \mathbb{R}.$$

□

(b) Find vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^4$ such that $\operatorname{Nul} A = \{r\mathbf{u} + s\mathbf{v} \mid r, s \in \mathbb{R}\}$.
Solution:

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

□

(52) Let $V := \{f : \mathbb{R} \to \mathbb{R}\}$ be the vector space of functions on $\mathbb{R}$.

(a) Is $\{f : \mathbb{R} \to \mathbb{R} \mid f(0) = 1\}$ a subspace of V?
Solution:
No. The zero vector is the zero function. But the zero function is not 1 at position 0. So the zero vector does not belong to this set. □

(b) Is $\{f : \mathbb{R} \to \mathbb{R} \mid f(1) = 0\}$ a subspace of V?
Solution:
Yes. (1) The set contains the zero function, which is the zero vector. (2) If two functions f and g are in the set, then $(f + g)(1) = f(1) + g(1) = 0 + 0 = 0$. Thus the function $f + g$ is also in the set. (3) If f is in the set and $r \in \mathbb{R}$, then $(rf)(1) = rf(1) = r0 = 0$. Thus the function rf is also in the set. □

(c) Is $\{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$ a subspace of V?
Solution:
Yes. (1) The zero function is continuous. (2) The sum of continuous functions is continuous. (3) The multiplication of a continuous function by a scalar is

continuous. ☐

(53) Is $\{\mathbf{0}\}$ a subspace of $\mathbb{R}^n$?
Solution:
Yes, all three conditions for being a subspace are fulfilled. ☐

(54) Are the vectors $\mathbf{u} = 1$, $\mathbf{v} = t$, $\mathbf{w} = t^2$ in the vector space $V := \{f : \mathbb{R} \to \mathbb{R}\}$ linearly independent?
Solution:
Yes. We solve the system $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$. This means we find all triples (x_1, x_2, x_3) such that

$$\forall t \in \mathbb{R} \colon x_1 \cdot 1 + x_2 \cdot t + x_3 \cdot t^2 = 0.$$

We pick values for t. For $t = 0$ we get

$$x_1 + 0 + 0 = 0.$$

Thus $x_1 = 0$. For $t = 1$ we get

$$0 + x_2 + x_3 = 0,$$

and for $t = 2$ we get

$$0 + 2x_2 + 4x_3 = 0.$$

This is a linear system with variables x_2, x_3. The only solution is $x_2 = x_3 = 0$. Since also $x_1 = 0$, the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent. ☐