Math 3130 - Assignment 5

Due February 19, 2016
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Please write problems (37), (38), (39) on a sheet of paper separate from the rest.

(37) Prove the following part of the Invertible Matrix Theorem: Let A be an $n \times n$-matrix. If $C \cdot A = I_n$ for some matrix C, then $A \cdot \mathbf{x} = \mathbf{0}$ has only the trivial solution.

Solution:
Assume $CA = I$.
Let $\mathbf{x} \in \mathbb{R}^n$ be a solution of $A\mathbf{x} = \mathbf{0}$.
We multiply $A\mathbf{x} = \mathbf{0}$ by C from the left and obtain $CA\mathbf{x} = C\mathbf{0}$.
Since $CA = I$ and $C\mathbf{0} = \mathbf{0}$, we have $I\mathbf{x} = \mathbf{0}$. Now $I\mathbf{x} = \mathbf{x}$ yields $\mathbf{x} = \mathbf{0}$.
So $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$. □

(38) Prove the following part of the Invertible Matrix Theorem: Let A be an $n \times n$-matrix. A is invertible iff A^T is invertible.

Solution:
($\Rightarrow$) Assume A has an inverse B. That is $AB = I$ and $BA = I$.
By transposing the first equation we get $(AB)^T = I^T$. So $B^T A^T = I$.
Similarly $BA = I$ yields $A^T B^T = I$.
Since $(B^T)A^T = I = A^T(B^T)$, the matrix B^T is the inverse for A^T. Thus A^T is invertible.
($\Leftarrow$) Next assume A^T is invertible and show A is invertible.
By the implication ($\Rightarrow$) that we just proved, we know that also $(A^T)^T$ is invertible.
Since $(A^T)^T = A$, the matrix A is invertible. □

(39) Assume that $T\colon \mathbb{R}^n \to \mathbb{R}^n, x \mapsto A \cdot x$ is bijective. Show that A is invertible.
Hint: Use that T is onto $\mathbb{R}^n$ and the Invertible Matrix Theorem.

Solution:
Assume T is bijective. That is T is onto and one-to-one.
Since T is onto $\mathbb{R}^n$, for every $b \in \mathbb{R}^n$ there is $x \in \mathbb{R}^n$ such that $T(x) = b$. But that means that $Ax = b$ has a solution for every $b \in \mathbb{R}^n$. By the Invertible Matrix Theorem this means that A is invertible. □

(40) Are the following matrices invertible? You do not need to compute the inverse. Just argue why or why not.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}, C = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 4 & 2 & 3 \end{bmatrix}$$

Solution:
A cannot be invertible because it is not sqare.
B is invertible because $1 \cdot 3 - (-2)2 \neq 0$ or because the columns of B are linearly independent ...
C is not invertible because it contains a 0-row. □

(41) Can a square matrix with 2 identical rows be invertible? Why or why not?
Solution:
No, if 2 rows are equal, you can subtract one from the other. Then an echelon form of the matrix will have a 0-row and cannot be transformed to the identity matrix. □

(42) Are the following mappings invertible? If so, give their inverses.
(a) $f\colon \mathbb{R} \to \mathbb{R}^2, x \mapsto \begin{bmatrix} 2x \\ 3x \end{bmatrix}$
(b) $g\colon \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x - 3y \\ -x + 2y \end{bmatrix}$
Solution:

(a) f is not invertible because it's not surjective. The image of f is just a line, not all of $\mathbb{R}^2$.
(b) g is invertible because its standard matrix $A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ has the inverse

$$A^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

Then $g^{-1}\colon \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto A^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$.

□

(43) Let T be the rotation of $\mathbb{R}^2$ around the origin by the angle φ counterclockwise. Is the standard matrix of T invertible? If so, write down a formula for T^{-1}. What is its geometric interpretation?
Solution:
The standard matrix of T is

$$A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

and has the inverse

$$A^{-1} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}.$$

Hence

$$T^{-1}\colon \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto A^{-1} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Geometrically, this is the rotation around the origin by the angle φ clockwise or $-\varphi$ counterclockwise. □

(44) Are the following true or false? Explain why.
(a) Assume A implies B and B implies C. Then A implies C.
Solution:
True. Assume A implies B and B implies C. By definition this means if A is true, then B is true and if B is true, then C is true. Hence if A is true, we get

that C is true. Thus A implies C. □

(b) A implies B and B implies A means that A is true whenever B is true, and A is false whenever B is false.
Solution:
True. B implies A means that A is true whenever B is true. On the other hand, if B is false, then A cannot be true by A implies B. So A is false. Hence A implies B and B implies A yields that A is true whenever B is true, and A is false whenever B is false.
Conversely assume that A is true whenever B is true, and A is false whenever B is false. Then B implies A and $\neg B$ implies $\neg A$. The latter is the same as A implies B by contraposition. □

(c) n is an even integer $\Leftrightarrow n+1$ is an odd integer
Solution:
True. If n is even, then $n+1$ is odd and conversely. □

(d) For $x, y \in \mathbb{R}$, $xy = 0$ iff $x = 0$ and $y = 0$.
Solution:
False. If $xy = 0$, then $x = 0$ OR $y = 0$ but not necessarily both. □

(45) Give the negations of the following statements:
(a) $A \Rightarrow B$
(b) If you do well on the homework, you'll pass the class.
(c) $A \Leftrightarrow B$
(d) $x \in \mathbb{R}$ has an inverse if and only if $x \neq 0$.
Solution:

(a) $A \wedge \neg B$
(b) You do well on the homework and still won't pass the class.
(c) $A \Leftrightarrow \neg B$, $\neg A \Leftrightarrow B$, $(A \wedge \neg B) \vee (\neg A \wedge B)$
(d) $x \in \mathbb{R}$ has an inverse if and only if $x = 0$.

□