Math 3130 - Assignment 4

Due February 12, 2016
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(28) Let $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $T\colon \mathbb{R}^3 \to \mathbb{R}^2$, $\mathbf{x} \mapsto \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \mathbf{x}$.

(a) (3 points) Find the solution set of $T(\mathbf{x}) = \mathbf{b}$ in parametric vector form.

(b) (2 points) Give 2 vectors in $\mathbb{R}^3$ which are mapped to $\mathbf{b}$ by T, and give 2 vectors in $\mathbb{R}^3$ which are not mapped to $\mathbf{b}$ by T.

Solution:

(a)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 2 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

The variable x_3 is free and we obtain the solution

$$\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \qquad r \in \mathbb{R}$$

(b) E.g. $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are mapped to $\mathbf{b}$, and $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are not mapped to $\mathbf{b}$.

□

(29) Let T be as in (28) and let A be the standard matrix of T.

(a) Do the colums of A span $\mathbb{R}^2$?

(b) Are the columns of A linearly independent?

(c) Is T injective? Is T surjective? Is T bijective?

Solution:

The reduced echelon form of A is

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}.$$

(a) Yes since there is no zero row in echelon form.

(b) No since the system $Ax = 0$ has free variables (not every column has a pivot).

(c) From (a) we know that T is surjective, and from (b) we know that T is not injective. Thus T is not bijective.

□

(30) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear map such that

$$T(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, T(\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

(a) Express $\begin{bmatrix}1\\1\\2\end{bmatrix}$ as linear combination of $\begin{bmatrix}1\\2\\-1\end{bmatrix}$ and $\begin{bmatrix}2\\3\\1\end{bmatrix}$.

(b) Use the linearity of T to find $T(\begin{bmatrix}1\\1\\2\end{bmatrix})$ and $T(\begin{bmatrix}0\\1\\-3\end{bmatrix})$.

Solution:

(a) (1 point)

$$\begin{bmatrix}1 & 2 & 1\\2 & 3 & 1\\-1 & 1 & 2\end{bmatrix} \sim \cdots \sim \begin{bmatrix}1 & 0 & -1\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$$

The solution is $(-1, 1)$. Thus $\begin{bmatrix}1\\1\\2\end{bmatrix} = -\begin{bmatrix}1\\2\\-1\end{bmatrix} + \begin{bmatrix}2\\3\\1\end{bmatrix}$.

(b) (4 points) Similar to (a) we solve a system

$$\begin{bmatrix}1 & 2 & 0\\2 & 3 & 1\\-1 & 1 & -3\end{bmatrix} \sim \cdots \sim \begin{bmatrix}1 & 0 & 2\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}$$

and obtain a solution $(2, -1)$. Thus $\begin{bmatrix}0\\1\\-3\end{bmatrix} = 2\begin{bmatrix}1\\2\\-1\end{bmatrix} - \begin{bmatrix}2\\3\\1\end{bmatrix}$.

Now

$$T(\begin{bmatrix}1\\1\\2\end{bmatrix}) = -T(\begin{bmatrix}1\\2\\-1\end{bmatrix}) + T(\begin{bmatrix}2\\3\\1\end{bmatrix}) = -\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}-3\\-1\end{bmatrix},$$

$$T(\begin{bmatrix}0\\1\\-3\end{bmatrix}) = 2T(\begin{bmatrix}1\\2\\-1\end{bmatrix}) - T(\begin{bmatrix}2\\3\\1\end{bmatrix}) = 2\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}4\\3\end{bmatrix}.$$

□

(31) Let $r, s \in \mathbb{R}$ and A, B be $m \times n$ matrices. Complete the proof of the following statement:

$$(r+s)(A+B) = rA + sA + rB + sB$$

Proof. Let $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Then

$((r+s)(A+B))_{ij}$	$=$	$\ldots$	by rule (b)
	$=$	$\ldots$	by rule (a)
	$=$	$\ldots$	by distributivity of real numbers
	$=$	$\ldots$	by rule (b)
	$=$	$(rA + sA + rB + sB)_{ij}$	by rule (a)

Thus $(r+s)(A+B) = rA + sA + rB + sB$. □

For the proof only use the following rules:

(a) $(C+D)_{ij} = C_{ij} + D_{ij}$ for all matrices C and D of the same size,
(b) $(tC)_{ij} = tC_{ij}$ for every scalar t and every matrix C.

Solution:

$$
\begin{aligned}
((r+s)(A+B))_{ij} &= (r+s)(A+B)_{ij} && \text{by rule (b)}\\
&= (r+s)(A_{ij}+B_{ij}) && \text{by rule (a)}\\
&= rA_{ij} + sA_{ij} + rB_{ij} + sB_{ij} && \text{by distrubutivity of real numbers}\\
&= (rA)_{ij} + (sA)_{ij} + (rB)_{ij} + (sB)_{ij} && \text{by rule (b)}\\
&= (rA + sA + rB + sB)_{ij} && \text{by rule (a)}
\end{aligned}
$$

□

(32) Let

$$
\mathbf{a}_1 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \ \mathbf{a}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \ \mathbf{a}_4 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}
$$

(a) Find $x_1, x_2, x_3, x_4 \in \mathbb{R}$ such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 = \mathbf{b}$. Verify your solution.
(b) Let $: \mathbb{R}^3 \to \mathbb{R}$ be a linear mapping such that $T(\mathbf{a}_1) = 10$, $T(\mathbf{a}_2) = 6$, $T(\mathbf{a}_3) = 8$, $T(\mathbf{a}_4) = 26$. Compute $T(\mathbf{b})$.

Solution:

(a) (4 points) We reduce the augmented matrix and obtain

$$
\begin{bmatrix} 2 & 1 & 0 & 3 & 1 \\ 0 & -1 & 2 & -1 & 1 \\ 2 & 0 & 1 & 1 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}.
$$

The variable x_4 is free, we choose some value, i.e. $x_4 = 0$. In this case we obtain $(-1, 3, 2, 0)$ as possible solution. Thus

$$
\mathbf{b} = -\mathbf{a}_1 + 3\mathbf{a}_2 + 2\mathbf{a}_3.
$$

(There are infinitely many possible linear combinations which yield $\mathbf{b}$.)
(b) (1 point) $T(\mathbf{b}) = -T(\mathbf{a}_1) + 3T(\mathbf{a}_2) + 2T(\mathbf{a}_3) = -10 + 3 \cdot 6 + 2 \cdot 8 = 24$.

□

(33) Let 0 be the 0-matrix with size 2×2. Find 2×2 matrices $A \neq 0$ and $B \neq 0$ such that $AB = 0$.

Solution:

E.g. $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

□

(34) (a) If possible, invert the following matrices:

$$
A = \begin{bmatrix} 2 & -3 \\ 4 & -9 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 2 \\ 4 & -3 \end{bmatrix}
$$

Solution:
$A^{-1} = \frac{1}{-18+12}\begin{bmatrix} -9 & 3 \\ -4 & 2 \end{bmatrix}$.
$B^{-1} = \begin{bmatrix} -3 & -2 \\ -4 & -3 \end{bmatrix}$.

□

(b) For which $a \in \mathbb{R}$ can the following matrix be inverted? Compute the inverse of C.

$$C = \begin{bmatrix} a-2 & 1 \\ 4 & a \end{bmatrix}$$

Solution:
For $(a-2)a - 4 = a^2 - 2a - 4 = 0$ the inverse is undefined. This is the case for $a = 1 \pm \sqrt{5}$. For other a the inverse is given by $\frac{1}{a^2-2a-4}\begin{bmatrix} a & -1 \\ -4 & a-2 \end{bmatrix}$. □

(35) If possible, invert the following matrix:

$$D = \begin{bmatrix} -3 & 2 & 4 \\ 0 & 1 & -2 \\ 1 & -3 & 4 \end{bmatrix}$$

Solution:

$$\left[\begin{array}{cccccc} -3 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 1 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 10 & 4 \\ 0 & 1 & 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & 1/2 & 7/2 & 3/2 \end{array}\right]$$

The inverse is the rightmost 3×3 block. □

(36) If possible, invert the following matrix:

$$E = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ -1 & 2 & -1 \end{bmatrix}$$

Solution:

$$\left[\begin{array}{cccccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cccccc} 1 & 0 & 3 & 0 & 2 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

There is no inverse. (It's not necessary to complete the reduction.) □

References

[1] David C. Lay. Linear Algebra and Its Applications. Addison-Wesley, 4th edition, 2012.