Math 3130 - Assignment 3

Due February 5, 2016
Markus Steindl

(19) Show that the following maps are not linear by giving concrete vectors for which the defining properties of linear maps are not satisfied.

(a) $g : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} xy \\ y \end{bmatrix}$

(b) $h : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} |x| + |y| \\ 2x \end{bmatrix}$

Solution:
For example

(a) $g(2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \cdot g(\begin{bmatrix} 1 \\ 1 \end{bmatrix})$

(b) $h((-1) \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \neq \begin{bmatrix} -1 \\ -2 \end{bmatrix} = (-1) \cdot h(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$

□

(20) [1, Section 1.8, Ex 24] An *affine transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ has the form $T(\mathbf{x}) = A\mathbf{x} + b$ with A an $m \times n$-matrix and $b \in \mathbb{R}^m$. Show that T is not a linear transformation if $b \neq 0$.

Solution:
Let $\mathbf{0}$ denote the 0-vector in $\mathbb{R}^n$. Then $T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}) = b$ but $T(\mathbf{0}) + T(\mathbf{0}) = 2b$. Note that $b = 2b$ iff $b = \mathbf{0}$. Hence T is not linear if $b \neq \mathbf{0}$. □

(21) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map such that

$$T(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}, T(\begin{bmatrix} 3 \\ 2 \end{bmatrix}) = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

(a) Use the linearity of T to find $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$ and $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$.

(b) Determine $T(\begin{bmatrix} x \\ y \end{bmatrix})$ for arbitrary $x, y \in \mathbb{R}$.

Solution:

(a) First write the unit vectors as linear combinations of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Solve

$$x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

to get $x = -\frac{1}{2}$ and $y = \frac{1}{2}$. By the linearity of T we obtain

$$\begin{aligned} T(\begin{bmatrix}1\\0\end{bmatrix}) &= T(-\frac{1}{2}\begin{bmatrix}1\\2\end{bmatrix} + \frac{1}{2}\begin{bmatrix}3\\2\end{bmatrix}) \\ &= -\frac{1}{2}T(\begin{bmatrix}1\\2\end{bmatrix}) + \frac{1}{2}T(\begin{bmatrix}3\\2\end{bmatrix}) \\ &= -\frac{1}{2}\begin{bmatrix}2\\0\\-3\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-2\\2\\1\end{bmatrix} \\ &= \begin{bmatrix}-2\\1\\2\end{bmatrix} \end{aligned}$$

Similarly we compute that

$$\begin{bmatrix}0\\1\end{bmatrix} = \frac{3}{4}\begin{bmatrix}1\\2\end{bmatrix} - \frac{1}{4}\begin{bmatrix}3\\2\end{bmatrix}$$

and hence obtain

$$T(\begin{bmatrix}0\\1\end{bmatrix}) = \frac{3}{4}\begin{bmatrix}2\\0\\-3\end{bmatrix} - \frac{1}{4}\begin{bmatrix}-2\\2\\1\end{bmatrix} = \begin{bmatrix}2\\-1/2\\-5/2\end{bmatrix}$$

(b) By (a) we know the standard matrix of T is

$$A = \begin{bmatrix}-2 & 2\\1 & -1/2\\2 & -5/2\end{bmatrix}.$$

Thus $T(\begin{bmatrix}x\\y\end{bmatrix}) = A \cdot \begin{bmatrix}x\\y\end{bmatrix}$.

□

(22) Give the standard matrices for the following linear transformations:

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$, $\begin{bmatrix}x\\y\end{bmatrix} \mapsto \begin{bmatrix}2x+y\\x\\-x+y\end{bmatrix}$

Solution:
Just take the coefficient matrix of the transformation to get its standard matrix

$$A = \begin{bmatrix}2 & 1\\1 & 0\\-1 & 1\end{bmatrix}$$

□

(b) the function S on $\mathbb{R}^2$ that scales all vectors to half their length.

Solution:
The function is $S : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix}x\\y\end{bmatrix} \mapsto \frac{1}{2}\begin{bmatrix}x\\y\end{bmatrix}$ and has standard matrix $A = \begin{bmatrix}\frac{1}{2} & 0\\0 & \frac{1}{2}\end{bmatrix}$.

□

(23) Give the standard matrix for the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates points (about the origin) by 60° counterclockwise and then reflects them on the x-axis.

Solution:

The rotation maps $\mathbf{e}_1$ to $\begin{bmatrix} \cos 60 \\ \sin 60 \end{bmatrix}$, which is mapped to $\begin{bmatrix} \cos 60 \\ -\sin 60 \end{bmatrix}$ by the reflection. Similarly $\mathbf{e}_2$ is rotated to $\begin{bmatrix} -\sin 60 \\ \cos 60 \end{bmatrix}$ and then reflected to $\begin{bmatrix} -\sin 60 \\ -\cos 60 \end{bmatrix}$.

So the standard matrix of T is

$$A = \begin{bmatrix} \cos 60 & -\sin 60 \\ -\sin 60 & -\cos 60 \end{bmatrix}$$

.

□

(24) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection at the line $2x + 3y = 0$. Note that T is linear.

(a) What is the reflection of the normal vector $\mathbf{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ of the line? What is the reflection of the vector $\mathbf{b} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, which is on this line? Make a drawing if necessary.

(b) Write the unit vectors $\mathbf{e}_1, \mathbf{e}_2$ as linear combinations of $\mathbf{a}$ and $\mathbf{b}$.

(c) Use the linearity of T to find the reflection of the unit vectors $T(\mathbf{e}_1), T(\mathbf{e}_2)$ from $T(\mathbf{a}), T(\mathbf{b})$.

(d) Give the standard matrix for T.

Solution:

(a) $T(\mathbf{a}) = -\mathbf{a}$ and $T(\mathbf{b}) = \mathbf{b}$.

(b) Solve $x\mathbf{a} + y\mathbf{b} = \mathbf{e}_1$. From

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 1 \\ 0 & -\frac{13}{2} & -\frac{3}{2} \end{bmatrix}$$

we get $x = \frac{2}{13}, y = \frac{3}{13}$.

So $\mathbf{e}_1 = \frac{2}{13}\mathbf{a} + \frac{3}{13}\mathbf{b}$. Similarly we find $\mathbf{e}_2 = \frac{3}{13}\mathbf{a} - \frac{2}{13}\mathbf{b}$.

(c) Since T is linear, (b) yields

$$T(\mathbf{e}_1) = \frac{2}{13}T(\mathbf{a}) + \frac{3}{13}T(\mathbf{b}) = -\frac{2}{13}\mathbf{a} + \frac{3}{13}\mathbf{b} = \begin{bmatrix} 5/13 \\ -12/13 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \frac{3}{13}T(\mathbf{a}) - \frac{2}{13}T(\mathbf{b}) = -\frac{3}{13}\mathbf{a} - \frac{2}{13}\mathbf{b} = \begin{bmatrix} -12/13 \\ -5/13 \end{bmatrix}$$

(d) By (c) the standard matrix of T is

$$A = \frac{1}{13}\begin{bmatrix} 5 & -12 \\ -12 & -5 \end{bmatrix}$$

□

(25) Is

$$T : \mathbb{R}^3 \to \mathbb{R}^2, x \mapsto \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix} \cdot x$$

injective, surjective, bijective?

Solution:

Not injective because x_1 is free in $A \cdot \mathbf{x} = \mathbf{0}$. Alternatively, the columns of A are linearly dependent. So T is not injective (Theorem 12 of Section 1.9).

Surjective because A is in row echelon form and has no 0-rows (Theorem 12 of Section 1.9).

Bijective means injective and surjective. Hence T is not bijective because it is not injective. □

(26) [1, cf. Section 1.9, Ex 23/24] True or False? Correct the false statements to make them true.

(a) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by the images of the unit vectors in $\mathbb{R}^n$.

(b) $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if every vector $\mathbf{x} \in \mathbb{R}^n$ is mapped onto some vector in $\mathbb{R}^m$.

(c) $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if every vector $\mathbf{x} \in \mathbb{R}^n$ is mapped onto a unique vector in $\mathbb{R}^m$.

(d) A linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ cannot be one-to-one.

Solution:

(a) True because every vector is a linear combination of unit vectors.

(b) False. Any function $T : \mathbb{R}^n \to \mathbb{R}^m$ maps every vector $\mathbf{x} \in \mathbb{R}^n$ onto some vector $T(\mathbf{x})$ in $\mathbb{R}^m$.
The correct statement is: $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if for every vector $\mathbf{y} \in \mathbb{R}^m$ there is some vector $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{y}$.

(c) False. Any function $T : \mathbb{R}^n \to \mathbb{R}^m$ maps every vector $\mathbf{x} \in \mathbb{R}^n$ onto the unique vector $T(\mathbf{x})$ in $\mathbb{R}^m$.
The correct statement is: $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if any 2 distinct vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ are mapped to distinct vectors $T(\mathbf{x}_1), T(\mathbf{x}_2)$.

(d) True because when solving $A \cdot \mathbf{x} = \mathbf{0}$ for a 2×3-matrix A there will be at least one free variable.

□

(27) Compute if possible

$$A + 3B, B \cdot A, A \cdot B, A \cdot C, C \cdot A$$

for the matrices

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix}, C = \begin{bmatrix} -1 & 2 \\ 0 & 4 \\ 1 & 3 \end{bmatrix}.$$

If an expression is undefined, explain why.

Solution:

$A + 3B$ is undefined because A has more columns than B.

$$B \cdot A = \begin{bmatrix} 9 & 12 & 3 \\ -4 & -9 & -2 \end{bmatrix}$$

$A \cdot B$ is undefined because the rows of A are longer than the columns of B.

$$A \cdot C = \begin{bmatrix} -2 & 0 \\ -2 & 25 \end{bmatrix}$$

$$C \cdot A = \begin{bmatrix} 4 & 9 & 2 \\ 12 & 16 & 4 \\ 11 & 11 & 3 \end{bmatrix}$$

□

References

[1] David C. Lay. Linear Algebra and Its Applications. Addison-Wesley, 4th edition, 2012.