Math 3130 - Assignment 2

Due January 29, 2016
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(10) [1, Section 1.4, Ex 17] How many rows of A contain a pivot position? Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for each $\mathbf{b} \in \mathbb{R}^4$?

$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & -1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$$

Solution:

$$\sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 2 \\ 0 & -4 & 2 & -8 \\ 0 & -2 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(\sim \begin{bmatrix} 1 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

From the (reduced) echelon matrix we see that the last row has no pivot. By Theorem 4 the system $A\mathbf{x} = \mathbf{b}$ does not have a solution for each $\mathbf{b} \in \mathbb{R}^4$. $\square$

Grading: Alternatively, students may write down an augmented matrix with RHS $b_1, \ldots, b_4$ and reduce it to echelon form. The last row has then the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & c_1 b_1 + c_2 b_2 + c_3 b_3 + c_4 b_4 \end{bmatrix}$$

with $c_1, \ldots, c_4 \in \mathbb{R}$. Now one can see that the expression $c_1 b_1 + c_2 b_2 + c_3 b_3 + c_4 b_4$ can be nonzero for some $\mathbf{b}$. For such $\mathbf{b}$ there is no solution.

(11) [1, Section 1.4, Ex 31] Let A be a 3×2 matrix. Explain why the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b} \in \mathbb{R}^n$.
Solution:
The matrix A has at most 2 pivot positions. This means at least one row has no pivot position. By Theorem 4 the system is not consistent for all $\mathbf{b} \in \mathbb{R}^n$. $\square$

(12) Let $\mathbf{u} \in \mathbb{R}^n$ be a vector and let $c, d \in \mathbb{R}$ be scalars. Show that

$$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}.$$

Solution:

$$(c+d)\mathbf{u} = (c+d) \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} (c+d)u_1 \\ \vdots \\ (c+d)u_n \end{bmatrix} = \begin{bmatrix} cu_1 + du_1 \\ \vdots \\ cu_n + du_n \end{bmatrix} = c \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + d \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = c\mathbf{u} + d\mathbf{u}$$

$\square$

(13) [1, cf. Section 1.5, Ex 17] Let

$$A = \begin{bmatrix} 2 & 2 & 4 \\ -4 & -4 & -8 \\ 0 & -3 & -3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 8 \\ -16 \\ 12 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solve the equations $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$. Express both solution sets in parametric vector form. Give a geometric description of the solution sets.

Solution:

We solve $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 2 & 2 & 4 & 8 \\ -4 & -4 & -8 & -16 \\ 0 & -3 & -3 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 8 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The solution in parametric vector form is

$$\mathbf{x} = \begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix} + r \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad r \in \mathbb{R}.$$

The solution set is a line through the point $(8, -4, 0)$ spanned by the vector $(-1, -1, 1)$. For the homogeneous system $A\mathbf{x} = \mathbf{0}$ we obtain

$$\mathbf{x} = r \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad r \in \mathbb{R}.$$

This solution set is a line through the origin spanned by the vector $(-1, -1, 1)$. □

Grading: Some students may sketch the line instead of giving a text description, which is fine as well. The geometric interpretation is worth 0.5 points.

(14) [1, cf. Section 1.5, Ex 11] Let

$$A = \begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solve the equations $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$. Express both solution sets in parametric vector form.

Solution:

We solve $A\mathbf{x} = \mathbf{b}$. The augmented matrix

$$\begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -4 & 1 \end{bmatrix}$$

is already in echelon form. We reduce further and obtain

$$\sim \begin{bmatrix} 1 & -4 & -2 & 0 & 0 & 7 & -2 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -4 & 1 \end{bmatrix}$$

The free variables are x_2, x_4, x_6. We rename them to r, s, t and obtain

$$\begin{aligned} x_1 &= 4r - 5t \\ x_2 &= r \\ x_3 &= 1 + t \\ x_4 &= s \\ x_5 &= 1 + 4t \\ x_6 &= t. \end{aligned}$$

The solution in parametric vector form is

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{bmatrix}, \quad r, s, t \in \mathbb{R}.$$

For the homogeneous system $A\mathbf{x} = \mathbf{0}$ we obtain

$$\mathbf{x} = r \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{bmatrix}, \quad r, s, t \in \mathbb{R}.$$

□

(15) [1, Section 1.5, Ex 31] Let A be a 3×2 matrix with 2 pivot positions.
(a) Does the equation $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution?
(b) Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for every possible $\mathbf{b} \in \mathbb{R}^3$?
Explain your answers!
Solution:

(a) Both columns have a pivot. Thus there are no free variables, and the system has only the trivial solution.
(b) The last row does not have a pivot. Thus the answer is no by Theorem 4.

□

(16) [1, Section 1.7, Ex 9] Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}.$$

(a) For which values of h is $\mathbf{w}$ in Span$\{\mathbf{u}, \mathbf{v}\}$?
(b) For which values of h is $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ linearly dependent?
Solution:

(a) We solve $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{w}$. We reduce the augmented matrix to echelon form:

$$\begin{bmatrix} 1 & -3 & 5 \\ -3 & 9 & -7 \\ 2 & -6 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & 10-h \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

For no value of h does a solution exist.

(b) We solve $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$. We reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 5 & 0 \\ -3 & 9 & -7 & 0 \\ 2 & -6 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 10-h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The reduced system has free variables and does not depend on h. Thus there are nontrivial solutions, and hence $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent for every h.

□

(17) [1, cf. Section 1.7, Ex 21] Mark each statement True or False, and justify each answer.

(a) The columns of a matrix A are linearly independent if $\mathbf{x} = \mathbf{0}$ is a solution of $A\mathbf{x} = \mathbf{0}$.

(b) If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, then each vector is a linear combination of the other two vectors.

(c) The columns of any 4×5 matrix are linearly dependent.

(d) If $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, and if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, then $\mathbf{w}$ is in the span of $\mathbf{u}, \mathbf{v}$.

Solution:

(a) False. E.g. consider $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

(b) False. E.g. for $\mathbf{u} = \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{w}$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(c) True by Theorem 8.

(d) True. There are coefficients x_1, x_2, x_3, not all equal to zero, such that $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = 0$. If x_3 was 0, then we had $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{0}$, which is impossible. Thus $x_3 \neq 0$. So $\mathbf{w} = -\frac{x_1}{x_3}\mathbf{u} - \frac{x_2}{x_3}\mathbf{v}$.

□

(18) Show the following Theorem in 2 steps: Suppose $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{p}$. Then the set of all solutions of $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{p} + \operatorname{NullSpace} A = \{\mathbf{p} + \mathbf{v} \mid \mathbf{v} \in \operatorname{NullSpace} A\}.$$

Suppose $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{p}$.

(a) Show that if $\mathbf{v}$ is in $\operatorname{NullSpace} A$, then $\mathbf{p} + \mathbf{v}$ is also a solution for $A\mathbf{x} = \mathbf{b}$.

(b) Show that if $\mathbf{q}$ is a solution for $A\mathbf{x} = \mathbf{b}$, then $\mathbf{q} - \mathbf{p}$ is in $\operatorname{NullSpace} A$.

Solution:

(a) Assume $\mathbf{v}$ is in the null space of A. We know that $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{p} = \mathbf{b}$. Thus

$$A(\mathbf{p} + \mathbf{v}) = A\mathbf{p} + A\mathbf{v} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

Hence $\mathbf{p} + \mathbf{v}$ is a solution of $A\mathbf{x} = \mathbf{b}$.

(b) Assume $\mathbf{q}$ is a solution of $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{p} = \mathbf{b}$ and $A\mathbf{q} = \mathbf{b}$. Thus

$$\begin{aligned} A(\mathbf{q} - \mathbf{p}) &= A(\mathbf{q} + (-1)\mathbf{p}) = A\mathbf{q} + A(-1)\mathbf{p} = A\mathbf{q} + (-1)A\mathbf{p} \\ &= A\mathbf{q} - A\mathbf{p} = \mathbf{b} - \mathbf{b} = \mathbf{0}. \end{aligned}$$

Hence $\mathbf{q} - \mathbf{p}$ is in NullSpace A.

□

References

[1] David C. Lay. Linear Algebra and Its Applications. Addison-Wesley, 4th edition, 2012.